MOSTOW RIGIDITY FOR FUCHSIAN BUILDINGS

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Abstract. We show that if a homeomorphism between the ideal boundaries of two Fuchsian buildings preserves the combinatorial cross ratio almost everywhere, then it extends to an isomorphism between the Fuchsian buildings. It follows that Mostow rigidity holds for Fuchsian buildings: if a group acts properly and cocompactly on two Fuchsian buildings X and Y, then X and Y are equivariantly isomorphic.

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Contents

1. **Introduction.** The classical Mostow rigidity theorem states that if X and Y are two irreducible symmetric spaces with nonpositive sectional curvature and dimension > 2, and G is a group acting properly and cocompactly on both X and Y, then X and Y are equivariantly isometric (after rescaling of the metrics). In this paper we consider the same question for a class of singular spaces — Fuchsian buildings.

Let R be a convex compact polygon in the hyperbolic plane whose angles are of the form π/m, $m \in \mathbb{Z}$, $m \geq 2$. The group generated by the reflections about the edges of R is a Coxeter group W, and $W(R)$ is a tessellation of the hyperbolic plane. The hyperbolic plane equipped with such a tessellation is the Coxeter complex of W. A two dimensional hyperbolic building Δ with chamber R (see Section 2 for the definition) is a piecewise hyperbolic 2-complex which has nice local structure and contains plenty of subcomplexes isomorphic to the Coxeter complex. Each 2-cell of Δ is isometric to R and is called a chamber. Each subcomplex of Δ isomorphic to the Coxeter complex is called an apartment. In particular, any two chambers of Δ are contained in a common apartment. Δ with the path metric is a CAT(-1) space. Each geodesic in Δ is also contained in an apartment.

The vertex links of Δ are generalized polygons (see [R] or Section 2). There is a combinatorial map $f : \Delta \to R$ such that the restriction of f to each chamber is an isomorphism. A two dimensional hyperbolic building Δ is a *Fuchsian building* if for each edge e of R there is an integer $q_e \geq 2$ such that any edge of Δ that is mapped to e under f is contained in exactly $q_e + 1$ chambers.

Theorem 1.1. *Let Δ_1, Δ_2 be two Fuchsian buildings, and G a group acting properly and cocompactly on both Δ_1 and Δ_2. Then Δ_1 and Δ_2 are equivariantly isomorphic.*

One cannot claim that Δ_1 and Δ_2 are equivariantly isometric as in the classical case. Two convex compact polygons R_1 and R_2 in the hyperbolic plane are said to be *angle-preserving isomorphic* if there is a combinatorial isomorphism from R_1 to R_2 that preserves the angles at the vertices. Let Δ be a Fuchsian building with chamber R, and R' a polygon angle-preserving isomorphic to R. Then it is not hard to see that one obtains a Fuchsian building Δ' by replacing each chamber of Δ by a copy of R'. The building Δ' is clearly isomorphic to Δ, but is not isometric to Δ when R and R' are not isometric. A convex compact polygon P is called *normal* if it has an inscribed circle that touches all its edges. By a result in A. Beardon's book (see [Bea], theorem 7.16.2.) each convex compact polygon P is angle-preserving isomorphic to a unique normal polygon P'. Theorem 1.1 can be restated as follows.

Theorem 1.2. *Let Δ_1, Δ_2 be two Fuchsian buildings whose chambers are normal, and G a group acting properly and cocompactly on both Δ_1 and Δ_2. Then Δ_1 and Δ_2 are equivariantly isometric.*

Our proof of Theorem 1.1 uses D. Sullivan's approach ([S2]) to Mostow rigidity and M. Bourdon's work ([Bo1], [Bo2]) on Fuchsian buildings. D. Sullivan's approach has also been used by M. Bourdon ([Bo1]) to establish Mostow rigidity for right angled Fuchsian buildings.

There are two steps in D. Sullivan's approach. First notice that under the conditions of Theorem 1.1, there is an equivariant homeomorphism $h : \partial\Delta_1 \to \partial\Delta_2$ between the ideal boundaries. In the first step one needs to show that there are conformal measures on $\partial\Delta_1$ and $\partial\Delta_2$ such that they have the same dimension and h is nonsingular with respect to the conformal measures(that is, $A \subset \partial\Delta_1$ has measure 0 if and only if $h(A)$ has measure 0). D. Sullivan's ergodic argument then shows that h preserves the cross ratio almost everywhere with respect to the conformal measures. In the second step one needs to establish the implication "h preserves the cross ratio almost everywhere" $\Longrightarrow$ "h extends to an isomorphism from Δ_1 to Δ_2". Step one has been established by M. Bourdon([Bo2]). He constructed combinatorial metrics on the ideal boundaries of Fuchsian buildings. The conformal measures on the ideal boundaries are the Hausdorff measures of the combinatorial metrics. M. Bourdon computed the Hausdorff dimension of the combinatorial metric and showed that the combinatorial metric realized the conformal dimension. In this paper we establish the second step.

Theorem 1.3. *Let Δ_1, Δ_2 be two Fuchsian buildings, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. Then h extends to an isomorphism from Δ_1 to Δ_2, that is, there is an isomorphism $f : \Delta_1 \to \Delta_2$ such that h agrees with the boundary map of f.*

There is a much stronger rigidity problem concerning Fuchsian buildings. In fact M. Bourdon and H. Pajot ([BP2]) have the following quasi-isometric rigidity conjecture.

Conjecture 1.4. *Let Δ_1, Δ_2 be two Fuchsian buildings whose chambers are normal. Then any quasi-isometry $f : \Delta_1 \to \Delta_2$ lies at a finite distance from an isometry.*

M. Bourdon and H. Pajot ([BP2]) have established this conjecture when the chambers of Δ_1, Δ_2 are right angled regular polygons. Recently B. Kleiner has informed the author of his new result on the quasiconformal group of a Q-regular, Q-Loewner space. By an argument of M. Bourdon and H. Pajot ([BP2]), this result and Theorem 1.3 imply that Conjecture 1.4 holds.

We next explain the proof of Theorem 1.3.

Let $h : \partial\Delta_1 \to \partial\Delta_2$ be a homeomorphism that preserves the combinatorial cross ratio (see Section 3.3 for definition) almost everywhere. It is not hard to show that for ξ, η in the ideal boundary of a Fuchsian building Δ, whether $\xi\eta$ is contained in the 1-skeleton can be detected by the behavior of the combinatorial cross ratio (see Lemma 3.11). It follows that if $\xi\eta$ $(\xi, \eta \in \partial\Delta_1)$ is a geodesic contained in the 1-skeleton of Δ_1, then $h(\xi)h(\eta)$ is contained in the 1-skeleton of Δ_2. We call $h(\xi)h(\eta)$ the image of $\xi\eta$. The key is to show that for any fixed vertex $v \in \Delta_1$, the images of all the geodesics in the 1-skeleton of Δ_1 through v intersect in a unique vertex w in Δ_2. Then it is not hard to see that the map $v \to w$ is a 1–to–1 map between the vertices that extend to an isomorphism from Δ_1 to Δ_2. One first needs to show that if c_1, c_2 are two intersecting geodesics contained in the 1-skeleton of an apartment of Δ_1 then their images intersect in a vertex.

Let Δ be a Fuchsian building, and $\eta \in \partial\Delta$ a point represented by a ray $\sigma : [0, \infty) \to \Delta$ contained in the 1-skeleton. Let $\xi_1, \xi_2 \in \Delta \cup \partial\Delta - \{\eta\}$ be such that $\xi_1\eta \cap \sigma = \xi_2\eta \cap \sigma = \phi$. We parameterize $\xi_i\eta$ $(i = 1, 2)$ by $\sigma_i : I_i \to \Delta$ $(I_i \subset (-\infty, \infty))$ such that $\sigma_i(t)$ and $\sigma(t)$ lie on the same horosphere centered at η. Since Δ is CAT(-1), we have $d(\sigma_i(t), \sigma(t)) \to 0$ as $t \to \infty$. We say ξ_1 and ξ_2 *lie at different sides of* η if $\sigma_1(t)$ and $\sigma_2(t)$ lie in different chambers for all t.

Definition 1.5. Let $\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$ be pairwise distinct such that the geodesics $\xi_1\xi_2$, $\eta_1\eta_2$ are contained in the 1-skeleton. We say the two geodesics $\xi_1\xi_2$, $\eta_1\eta_2$ *are at different sides* if ξ_1, ξ_2 lie at different sides of η_i $(i = 1, 2)$ and η_1, η_2 lie at different sides of ξ_i $(i = 1, 2)$.

A crucial lemma (see Lemma 3.12) says that for $\xi_1, \xi_2 \in \partial\Delta$ and $\eta \in \partial\Delta$ represented by a geodesic ray $\sigma : [0, \infty) \to \Delta$ contained in the 1-skeleton with $\xi_1\eta \cap \sigma = \xi_2\eta \cap \sigma = \phi$, whether ξ_1, ξ_2 lie at different sides of η can be detected by the combinatorial cross ratio. It follows that for $\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta_1$, the two geodesics $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides if and only if their images are at different sides. In particular it implies that the images of two intersecting geodesics contained in the 1-skeleton of an apartment of Δ_1 are at different sides. It turns out that in most cases, two geodesics at different sides must intersect in a point.

Let us see what happens if two geodesics at different sides are disjoint. Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two such geodesics. Continuity argument shows that for each $i = 1, 2$ there are $x_i \in \xi_1\xi_2$, $y_i \in \eta_1\eta_2$ such that $x_i\eta_i \cap \eta_1\eta_2$ and $y_i\xi_i \cap \xi_1\xi_2$ are geodesic rays. This gives rise to triangles and quadrilaterals that are contained in the 1-skeleton. Hence we are led to consider triangles and quadrilaterals that are contained in the 1-skeleton. Each such triangle or quadrilateral must bound a topological disk which is the union of a finite number of chambers (see Section 5). Then Gauss-Bonnet implies that there are only a few such triangles and quadrilaterals. For instance, there is no such triangle or quadrilateral when the chamber of Δ has at least 5 edges, that is, $k \geq 5$. Hence in this case two geodesics at different sides must intersect in a point. The same is true when the chamber is not a right triangle. The right triangle case is more involved, and it is indeed possible to have disjoint geodesics that are at different sides. But two disjoint geodesics that are at different sides in Δ_2 cannot occur as the images of two intersecting geodesics contained in a common apartment of Δ_1. For the proof one needs to use some special properties of vertex links, which are generalized polygons.

The proof of Theorem 1.3 is relatively simple when the chambers of Δ_1 and Δ_2 have at least 5 edges, please see Remark 6.5.

The paper is organized as follows. In Section 2 we recall the definition of Fuchsian buildings and some facts concerning generalized polygons. In Section 3 we recall combinatorial cross ratio defined on the ideal boundary of a Fuchsian building, and show that two important geometric properties can be detected by the behavior of the combinatorial cross ratio. In Section 4 we review D. Sullivan's approach to Mostow rigidity and M. Bourdon's work on Fuchsian buildings. In Section 5 we state the main facts about triangles and quadrilaterals that are contained in the 1-skeleton. The proofs of these results are contained in the last section (Section 10) to prevent a major disruption of the exposition. In Section 6 we give sufficient conditions for two geodesics to be at different sides and study the intersection of two geodesics at different sides. In Sections 7-9 we use the tools developed in the previous sections to show that the images of all the geodesics contained in the 1-skeleton of

Δ_1 and through a fixed vertex of Δ_1 intersect in a unique vertex. The proof is divided into three cases: when the chambers are not right triangles (Section 7); the chambers are right triangles but different from $(2,3,8)$ (Section 8); the chambers are $(2,3,8)$ (Section 9).

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2. **Fuchsian buildings.** In this section we define Fuchsian buildings and review some facts about generalized polygons. Fuchsian buildings were introduced by M. Bourdon ([Bo2]). Our presentation closely follows [Bo2].

2.1. *Convex polygons and reflections groups in* $\mathbb{H}^2$. In this paper, R denotes a compact convex polygon in $\mathbb{H}^2$, whose angles are of the form π/m, $m \in \mathbb{N}$, $m \geq 2$, and W the Coxeter group generated by the reflections about the edges of R (see [M], theorem IV.H.11).

We label the edges of R cyclically by $\{1\}$, $\{2\}$, $\cdots$, $\{k\}$ and the vertices by $\{1,2\}, \cdots$, $\{k-1,k\}, \{k,1\}$ such that the edges i and $i+1$ intersect at the vertex $\{i,i+1\}$. The angle of R at the vertex $\{i,i+1\}$, $i \in \mathbb{Z}/k\mathbb{Z}$, is denoted by

$$\alpha_{i,i+1} = \pi/m_{i,i+1}, \text{with } \ m_{i,i+1} \in \mathbb{N}, \ m_{i,i+1} \geq 2.$$

It is well-known that (see [M], theorem IV.H.11) the images of R under W form a tessellation of $\mathbb{H}^2$ and the quotient $\mathbb{H}^2/W$ equals R. It follows that there is a labeling of the edges and vertices of the tessellation that is W-invariant and compatible with that of R. We let A_R be the obtained labeled 2-complex.

Two convex compact polygons P_1 and P_2 in $\mathbb{H}^2$ are *angle-preserving isomorphic* if there is a combinatorial isomorphism from P_1 to P_2 that preserves the angles at the vertices. A convex compact polygon P is called *normal* if it has an inscribed circle that touches all its edges. By a result in A. Beardon's book (see [Bea], theorem 7.16.2.) each convex compact polygon P is angle-preserving isomorphic to a unique normal polygon P'. Notice that all compact triangles in $\mathbb{H}^2$ are normal. We shall use (m_1,m_2,m_3) to denote the triangle (unique up to isometry) with angles π/m_1, π/m_2 and π/m_3.

2.2. *2-dimensional hyperbolic buildings.* Let R be a fixed polygon as defined in Section 2.1.

Definition 2.1. Let Δ be a connected cellular 2-complex, whose edges and vertices are labeled by $\{1\}$, $\{2\}$, $\cdots$, $\{k\}$ and $\{1,2\}, \cdots, \{k-1,k\}, \{k,1\}$ respectively, such that each 2-cell (called a chamber) is isomorphic to R as labeled 2-complexes. Δ is called a *2-dimensional hyperbolic building* if it has a family of subcomplexes (called apartments) isomorphic to A_R (as labeled 2-complexes) with the following properties:
(1) Given any two chambers, there is an apartment containing both;
(2) For any two apartments A_1, A_2 that share a chamber, there is an isomorphism of labeled 2-complexes $f: A_1 \to A_2$ which pointwise fixes $A_1 \cap A_2$.
Δ is called a *Fuchsian building* if in addition, there are integers $q_i \geq 2$, $i = 1,\ 2,\ \cdots, k$ such that each edge of Δ labeled by i is contained in exactly $q_i + 1$ chambers.

Let Δ be a 2-dimensional hyperbolic building, A an apartment and $C \subset A$ a chamber. The retraction onto A centered at C is a label-preserving cellular map $r_{A,C}: \Delta \to A$ defined as follows. For any chamber C' of Δ, choose an apartment A' containing both C and C'. Since $A \cap A'$ contains C, there is a label-preserving isomorphism $f: A' \to A$ that pointwise fixes $A \cap A' \supset C$. Set $r_{A,C}|_{C'} = f|_{C'}$. One checks that $r_{A,C}$ is well-defined.

Since a chamber is label-preserving isomorphic to R, there is a metric on each chamber making it isometric to R and we can glue the chambers together along the edges by isometries. We equip Δ with the path metric. Then Δ is a CAT(-1) space (see [BH] for definition) and in particular a Gromov hyperbolic space. The boundary at infinity $\partial\Delta$ is homeomorphic to the Menger curve (see

[Ben]). Each apartment is a convex subset of Δ. Since the limits of apartments are apartments, the local finiteness of Fuchsian buildings implies that any geodesic in a Fuchsian building is contained in an apartment.

Let Δ be a Fuchsian building. By Section 2.3 below, $m_{i,i+1} \in \{2,3,4,6,8\}$; $q_i = q_{i+1}$ if $m_{i,i+1} = 3$, and $q_i \neq q_{i+1}$ if $m_{i,i+1} = 8$.

Let Δ be a Fuchsian building, and $A \subset \Delta$ an apartment of Δ. A *wall* in A is a complete geodesic contained in the 1-skeleton of A. Let W be a wall of A, $v \in W$ a vertex and $e_1, e_2 \subset W$ the two edges of W containing v. By assumption the vertex v is labeled by $\{i, i+1\}$ for some i. If $m_{i,i+1}$ is even, then e_1, e_2 are both labeled by i or both labeled by $i+1$. If $m_{i,i+1} = 3$, then one of e_1, e_2 is labeled by i, the other is labeled by $i+1$ and $q_i = q_{i+1}$. In any case, the number of chambers containing e_1 equals the number of chambers containing e_2. It follows that there is some $i, 1 \leq i \leq k$ such that each edge in W is contained in exactly $q_i + 1$ chambers. For any wall W and any edge $e \subset W$, we set $l(e) = l(W) = \log q_i$ if e is labeled by i.

Let C, C' be two chambers. A *gallery* from C to C' with length n is a sequence $\mathcal{G} = (C_0 = C, e_1, C_1, \cdots, C_{i-1}, e_i, C_i, \cdots, C_{n-1}, e_n, C_n = C')$, where each C_i is a chamber and e_{i-1} is a common edge of C_{i-1} and C_i. Notice that here we allow C_{i-1} and C_i to be the same. A gallery is minimal if it has the smallest length among all galleries from C to C'. We define $l(\mathcal{G}) = l(e_1) + \cdots + l(e_n)$.

Let A be an apartment and $C, C' \subset A$ two chambers. A wall $W \subset A$ is said to separate C and C' if the interiors of C and C' lie in different components of $A - W$. Let $\mathcal{W}_A(C, C')$ be the set of walls in A separating C and C'. Note that each gallery in A from C to C' must cross each wall in $\mathcal{W}_A(C, C')$ at least once. On the other hand, it is easy to find a gallery from C to C' that crosses each wall in $\mathcal{W}_A(C, C')$ exactly once as follows. Pick generic points $x \in \text{interior}(C)$, $y \in \text{interior}(C')$ such that xy does not contain any vertices. Then we obtain a gallery $\mathcal{G}_{x,y}$ from C to C' by recording in linear order the chambers and edges intersected by xy. Convexity implies that xy intersects each wall in $\mathcal{W}_A(C, C')$ at exactly one point, and so the gallery crosses each wall in $\mathcal{W}_A(C, C')$ exactly once.

Finally let us remark that there are uncountably many isomorphism classes of Fuchsian buildings ([GP]). There are also many constructions of Fuchsian buildings admitting proper and cocompact group actions: complexes of groups ([Bo2], [GP]), ramified coverings of Euclidean buildings, blueprints of Ronan-Tits ([RT]), etc.

2.3. *Generalized polygons and vertex links.* The reader is referred to [R] (chapter 3) and [T] (section 3) for more details on the material in this subsection.

Generalized polygons are also called rank two spherical buildings. Let L be a connected graph whose vertices are colored black and white such that the two vertices of each edge always have different colors. L is called a *generalized m-gon* ($m \in \mathbb{N}$, $m \geq 2$) if it has the following properties:
(1) Given any two edges, there is a circuit with combinatorial length $2m$ containing both;
(2) For two circuits A_1, A_2 of combinatorial length $2m$ that share an edge, there is an isomorphism $f: A_1 \to A_2$ that pointwise fixes $A_1 \cap A_2$.
When L is a generalized m-gon, an edge is called a chamber and a circuit with combinatorial length $2m$ is called an apartment. Two vertices of the same color are often said to have the same type. A theorem of Feit-Higman ([R], p.30) says that if L is a finite generalized m-polygon, then m belongs to $\{2,3,4,6,8\}$.

A generalized m-gon is *thick* if each vertex is contained in at least 3 chambers. Given a thick generalized m-gon L, there are integers $s, t \geq 2$ such that each black vertex is contained in exactly $s+1$ chambers and each white vertex is contained in exactly $t+1$ chambers (see p.29 of [R]). In this case, we say L has parameters (s,t). It is known that $s = t$ when m is odd (p.29 of [R]), and $s \neq t$ when $m = 8$ (see [FH]).

Let L be a generalized m-gon. We put a metric on each edge such that it is isomeric to the closed interval with length π/m. Note that all the apartments have length π. We equip L with

the path metric. Then L becomes a CAT(1) space (see [BH] for definition). All the apartments are convex in L, that is, if A is an apartment and $x, y \in A$ with $d(x,y) < \pi$, then the geodesic segment xy also lies in A. It follows that if two apartments A_1, A_2 share a chamber, then either $A_1 = A_2$ or $A_1 \cap A_2$ is a segment.

An injective edge path with combinatorial length m in a generalized m-gon is called a half apartment.

Lemma 2.2. *Let L be a thick generalized m-gon, and A, A' two apartments in L. Then there is a finite sequence of apartments $A_0 = A$, $A_1, \cdots, A_n = A'$ such that $A_i \cap A_{i+1}$ $(0 \leq i \leq n-1)$ is a half apartment.*

Proof. Let $e \subset A$, $e' \subset A'$ be two chambers. By considering an apartment containing e and e', we may assume $A \cap A'$ is a segment with combinatorial length l, $1 \leq l \leq m$. If $l = m$, we are done. Assume $l < m$. We claim in this case there is an apartment A'' such that $A \cap A''$ and $A' \cap A''$ have combinatorial length $> l$. The lemma clearly follows from the claim. Let v_1, v_2 be the endpoints of the segment $A \cap A'$, $e_1 \subset A$ the edge of A incident to v_1 but not contained in $A \cap A'$ and $e_2 \subset A'$ the edge of A' incident to v_2 but not contained in $A \cap A'$. Let A'' be an apartment containing e_1 and e_2. The convexity of apartments implies $A \cap A' \subset A''$, and the claim follows.

□

Two vertices of a generalized m-gon are *opposite* if the combinatorial distance between them is m.

Proposition 2.3. ([T], p.57) *Let L be a thick generalized m-gon with $m = 3, 4$ or 5. Then for any apartment A, there is a vertex opposite to all the vertices in A of the same type.*

Proposition 2.4. ([T], p.55) *Let L be a thick generalized polygon. Then for any two vertices v_1, v_2 of the same type, there is a vertex $v \in L$ opposite to both v_1 and v_2.*

Let Δ be a Fuchsian building and $v \in \Delta$ a vertex labeled by $\{i, i+1\}$. The link $\mathrm{Link}(\Delta, v)$ is a graph defined as follows. The vertex set of $\mathrm{Link}(\Delta, v)$ is in 1-to-1 correspondence with the set of edges of Δ incident to v. Two vertices of $\mathrm{Link}(\Delta, v)$ are connected by an edge if the edges of Δ corresponding to the two vertices are contained in a common chamber of Δ. A vertex of $\mathrm{Link}(\Delta, v)$ is black if the corresponding edge in Δ is labeled by i, and is white if the corresponding edge in Δ is labeled by $i+1$. Then it follows from the definitions that $\mathrm{Link}(\Delta, v)$ is a finite thick generalized $m_{i,i+1}$-gon with parameters (q_i, q_{i+1}). Consequently, $m_{i,i+1} \in \{2, 3, 4, 6, 8\}$; $q_i = q_{i+1}$ if $m_{i,i+1} = 3$, and $q_i \neq q_{i+1}$ if $m_{i,i+1} = 8$.

3. **Combinatorial cross ratio.** In this section we study combinatorial cross ratio on the ideal boundary of a Fuchsian building, and discuss two geometric properties that can be detected by combinatorial cross ratio. Combinatorial cross ratio was introduced and studied by M. Bourdon ([Bo1], [Bo2]). Our definitions are slightly different from M. Bourdon's in that we only use vertices in the dual graph.

3.1. *Dual graph.* Let Δ be a Fuchsian building. The dual graph $\mathcal{G}_\Delta$ of Δ is a metric graph defined as follows. The vertex set of $\mathcal{G}_\Delta$ is in 1-to-1 correspondence with the set of chambers of Δ. Two vertices of $\mathcal{G}_\Delta$ are connected by an edge with length $\log q_i$ if the corresponding chambers in Δ share an edge labeled by i. We equip $\mathcal{G}_\Delta$ with the path metric, and denote the distance between $x, y \in \mathcal{G}_\Delta$ by $|x - y|$. $\mathcal{G}_\Delta$ is clearly quasi-isometric to Δ, hence is Gromov hyperbolic and $\partial \mathcal{G}_\Delta = \partial \Delta$.

Abusing notation, for any chamber C, we shall also use C to denote the corresponding vertex in $\mathcal{G}_\Delta$.

Lemma 3.1. *Let C, C' be two chambers and A an apartment containing both. Then $|C - C'| = \Sigma_{W \in \mathcal{W}_A(C,C')} l(W)$. Furthermore, if ω' is an edge path in $\mathcal{G}_\Delta$ from C to C' with length $> |C - C'|$, then length$(\omega') \geq |C - C'| + \log q_i$ for some i.*

Proof. Let ω be a shortest path in $\mathcal{G}_\Delta$ from C to C'. The path ω gives rise to a gallery $\mathcal{G}_\omega$ in Δ from C to C' with $l(\mathcal{G}_\omega) = |C - C'|$. Let $\mathcal{G}'_\omega = r_{A,C}(\mathcal{G}_\omega)$. Since $r_{A,C}$ is label-preserving, $\mathcal{G}'_\omega$ is a gallery in A from C to C' with $l(\mathcal{G}'_\omega) = l(\mathcal{G}_\omega)$. On the other hand, for generic points $x \in C$, $y \in C'$, xy does not contain any vertices and $\mathcal{G}_{x,y}$ is a minimal gallery in A from C to C'. It follows that $l(\mathcal{G}_{x,y}) \leq l(\mathcal{G}'_\omega)$. It is clear that $\mathcal{G}_{x,y}$ determines an edge path $\omega_{x,y}$ in $\mathcal{G}_\Delta$ from C to C' such that $\text{length}(\omega_{x,y}) = l(\mathcal{G}_{x,y})$. Now the minimality of ω implies that $|C - C'| = l(\mathcal{G}_{x,y}) = \Sigma_{W \in \mathcal{W}_A(C,C')} l(W)$.

Let ω' be an edge path in $\mathcal{G}_\Delta$ from C to C' with length $> |C - C'|$, and denote $\mathcal{G}_{A,\omega'} = r_{A,C}(\mathcal{G}_{\omega'})$. Then $\mathcal{G}_{A,\omega'}$ is a gallery in A from C to C' with

$$l(\mathcal{G}_{A,\omega'}) = l(\mathcal{G}_{\omega'}) = \text{length}(\omega') > |C - C'| = \Sigma_{W \in \mathcal{W}_A(C,C')} l(W).$$

Since $\mathcal{G}_{A,\omega'}$ crosses each wall in $\mathcal{W}_A(C, C')$ at least once, there is some wall W of A such that $l(\mathcal{G}_{A,\omega'})$ contains the term $l(W)$ in addition to all the terms in $\Sigma_{W \in \mathcal{W}_A(C,C')} l(W)$. Hence we have $l(\mathcal{G}_{A,\omega'}) \geq |C - C'| + l(W)$.

□

Lemma 3.2. *Let C, C' be two chambers in Δ, and $x \in$ interior(C), $y \in$ interior(C'). Let $C_1 = C, \cdots, C_n = C'$ be the sequence of chambers in linear order with $xy \cap$ interior$(C_i) \neq \phi$. Then the sequence $C_1 = C, \cdots, C_n = C'$ defines a discrete geodesic in $\mathcal{G}_\Delta$ in the following sense: $|C_{i_1} - C_{i_3}| = |C_{i_1} - C_{i_2}| + |C_{i_2} - C_{i_3}|$ for any $1 \leq i_1 \leq i_2 \leq i_3 \leq n$.*

Proof. Let A be an apartment containing C and C'. Then the convexity of apartments implies that $C_i \subset A$ for all i. By Lemma 3.1, we have $|C_{i_1} - C_{i_3}| = \Sigma_{W \in \mathcal{W}_{A(C_{i_1}, C_{i_3})}} l(W)$, $|C_{i_1} - C_{i_2}| = \Sigma_{W \in \mathcal{W}_{A(C_{i_1}, C_{i_2})}} l(W)$ and $|C_{i_2} - C_{i_3}| = \Sigma_{W \in \mathcal{W}_{A(C_{i_2}, C_{i_3})}} l(W)$. Since the geodesic xy intersects the interiors of C_{i_1}, C_{i_2} and C_{i_3}, we see $\mathcal{W}_{A(C_{i_1}, C_{i_3})}$ is the disjoint union of $\mathcal{W}_{A(C_{i_1}, C_{i_2})}$ and $\mathcal{W}_{A(C_{i_2}, C_{i_3})}$. The lemma follows.

□

3.2. *Combinatorial Gromov product.* Let C be a vertex of $\mathcal{G}_\Delta$. Recall the Gromov product of $x \in \mathcal{G}_\Delta$ and $y \in \mathcal{G}_\Delta$ based at C is defined by: $\{x|y\}_C = \frac{1}{2}\{|x - C| + |y - C| - |x - y|\}$. For $\xi, \eta \in \partial\Delta = \partial\mathcal{G}_\Delta$, the combinatorial Gromov product of ξ and η based at C is:

$$\{\xi|\eta\}_C = \sup \liminf_{i,j \to \infty} \{x_i|y_j\}_C,$$

where the supreme is taken over all sequences of vertices in $\mathcal{G}_\Delta$ with $\{x_i\} \to \xi$, $\{y_j\} \to \eta$.

A sequence $\{C_i\}_{i=1}^\infty \subset \mathcal{G}_\Delta$ of vertices is called a *geodesic sequence* starting from C_1 if there is a geodesic ray σ in Δ starting from the interior of C_1 such that $\{C_i\}$ is the sequence of chambers (in linear order) whose interiors have nonempty intersection with σ. For $\xi, \eta \in \partial\Delta$, the modified Gromov product of ξ and η based at C is:

$$\{\xi|\eta\}'_C = \sup \liminf_{i,j \to \infty} \{C_i|D_j\}_C,$$

where the supreme is taken over all geodesic sequences $\{C_i\} \to \xi$, $\{D_j\} \to \eta$ starting from C. By definition, $\{\xi|\eta\}'_C \leq \{\xi|\eta\}_C$ always holds.

A point $\xi \in \partial\Delta$ is called a *singular point* if it is represented by a ray contained in the 1-skeleton. Otherwise $\xi \in \partial\Delta$ is called a *regular point*. Let $B \subset \partial\Delta$ be the set of regular points in $\partial\Delta$. For $x \in \Delta$ and $\xi \in \partial\Delta$, let $c_{x,\xi} : [0, \infty) \to \Delta$ denote the geodesic ray from x to ξ. Since the angles of a chamber belong to $\{\pi/2, \pi/3, \pi/4, \pi/6, \pi/8\}$, it is not hard to see that any ray representing a regular point cannot lie in a small neighborhood of the 1-skeleton:

Lemma 3.3. *There is some $\epsilon_0 > 0$ depending only on Δ with the following property. For any $\xi \in B$, $x \in \Delta$ and any $a > 0$, $c_{x,\xi}([a, +\infty)) \cap (\Delta - N_{\epsilon_0}(\Delta^{(1)})) \neq \phi$, where $N_{\epsilon_0}(\Delta^{(1)})$ is the ϵ_0-neighborhood of the 1-skeleton $\Delta^{(1)}$ of Δ.*

□

Lemma 3.4. *Let $\xi \neq \eta \in \partial\Delta$ be regular points. If $\{C_i\} \to \xi$, $\{D_j\} \to \eta$ are geodesic sequences starting from C, then there is some $i_0 \geq 1$ such that $\{\xi|\eta\}'_C = \{C_i|D_j\}_C$ for all $i, j \geq i_0$.*

Proof. By Lemma 3.2 the sequences $\{C_i\}$, $\{D_j\}$ are discrete geodesics in $\mathcal{G}_\Delta$. Triangle inequality implies that $\{C_{i_1}|D_{j_1}\}_C \leq \{C_{i_2}|D_{j_2}\}_C$ for $i_1 \leq i_2$, $j_1 \leq j_2$. Suppose $\{C_{i_1}|D_{j_1}\}_C < \{C_{i_2}|D_{j_2}\}_C$ for some $i_1 \leq i_2$, $j_1 \leq j_2$. Let ω_1 be a shortest path in $\mathcal{G}_\Delta$ from C_{i_2} to C_{i_1}, ω_2 be a shortest path from C_{i_1} to D_{j_1} and ω_3 be a shortest path from D_{j_1} to D_{j_2}. Then length$(\omega_3 * \omega_2 * \omega_1) > |C_{i_2} - D_{j_2}|$. Now Lemma 3.1 implies that length$(\omega_3 * \omega_2 * \omega_1) \geq |C_{i_2} - D_{j_2}| + \log q_i$ for some i. Consequently $\{C_{i_2}|D_{j_2}\}_C \geq \{C_{i_1}|D_{j_1}\}_C + \log q_i$. It follows that there is some integer $a \geq 1$ such that $\liminf_{i,j\to\infty}\{C_i|D_j\}_C = \{C_i|D_j\}_C$ for all $i, j \geq a$.

Let $\{C'_i\} \to \xi$, $\{D'_j\} \to \eta$ be two arbitrary geodesic sequences starting from C. The above paragraph shows that there is an integer $b \geq 1$ such that $\liminf_{i,j\to\infty}\{C'_i|D'_j\}_C = \{C'_i|D'_j\}_C$ for all $i, j \geq b$. By the definition of geodesic sequences, there are points $x, x', y, y' \in$ interior(C) such that $\{C_i\}$, $\{C'_i\}$, $\{D_j\}$, and $\{D'_j\}$ are the sequences of chambers (in linear order) whose interiors have nonempty intersection with $c_{x,\xi}$, $c_{x',\xi}$, $c_{y,\eta}$ and $c_{y',\eta}$ respectively. We reparameterize the geodesics $c_{x,\xi}$, $c_{x',\xi}$, $c_{y,\eta}$ and $c_{y',\eta}$ such that $c_{x,\xi}(t)$, $c_{x',\xi}(t)$ lie on the same horosphere centered at ξ, and $c_{y,\eta}(t)$, $c_{y',\eta}(t)$ lie on the same horosphere centered at η. Since Δ is a CAT(-1) space, $d(c_{x,\xi}(t), c_{x',\xi}(t)) \to 0$ and $d(c_{y,\eta}(t), c_{y',\eta}(t)) \to 0$ as $t \to +\infty$. On the other hand, Lemma 3.3 implies that there are arbitrarily large t, t' with $c_{x,\xi}(t) \notin N_{\epsilon_0}(\Delta^{(1)})$ and $c_{y,\eta}(t') \notin N_{\epsilon_0}(\Delta^{(1)})$. It follows that there are $i \geq a$, $j \geq a$ and $i' \geq b$, $j' \geq b$ with $C_i = C'_{i'}$ and $D_j = D'_{j'}$. Consequently, $\liminf_{i,j\to\infty}\{C'_i|D'_j\}_C = \{C'_{i'}|D'_{j'}\}_C = \{C_i|D_j\}_C = \liminf_{i,j\to\infty}\{C_i|D_j\}_C$ and the lemma follows.

□

Lemma 3.5. *Let $\xi, \eta \in \partial\Delta$ be regular points. Then $\{\xi|\eta\}_C = \{\xi|\eta\}'_C$.*

Proof. Let $\{x_i\} \to \xi$, $\{y_j\} \to \eta$ be two arbitrary sequences of vertices. We claim that $\{x_i|y_j\}_C = \{\xi|\eta\}'_C$ for sufficiently large i and j. The lemma follows from the claim.

Fix some $x \in$ interior(C) and let $\{C_k\}$ and $\{D_l\}$ be the geodesic sequences corresponding to the rays $x\xi$ and $x\eta$ respectively. By Lemma 3.4, there is some $i_0 \geq 1$ such that $\{\xi|\eta\}'_C = \{C_k|D_l\}_C$ for all $k, l \geq i_0$. Since ξ is regular and the asymptotic distance between $x\xi$ and $\eta\xi$ is 0, Lemma 3.3 implies that there is some $k \geq i_0$ such that both $x\xi \cap (C_k - N_{\frac{\epsilon_0}{2}}(\Delta^{(1)}))$ and $\xi\eta \cap (C_k - N_{\frac{\epsilon_0}{2}}(\Delta^{(1)}))$ are nonempty, where ϵ_0 is as in Lemma 3.3. Similarly there is some $l \geq i_0$ such that both $x\eta \cap (D_l - N_{\frac{\epsilon_0}{2}}(\Delta^{(1)}))$ and $\xi\eta \cap (D_l - N_{\frac{\epsilon_0}{2}}(\Delta^{(1)}))$ are nonempty. It follows that for sufficiently large i, j (so that x_i is sufficiently close to ξ and y_j is sufficiently close to η), and any $x' \in x_i$, $y' \in y_j$, we have $xx' \cap$ interior$(C_k) \neq \phi$, $x'y' \cap$ interior$(C_k) \neq \phi$, $xy' \cap$ interior$(D_l) \neq \phi$, and $x'y' \cap$ interior$(D_l) \neq \phi$. Now Lemma 3.2 implies that $\{x_i|y_j\}_C = \{C_k|D_l\}_C = \{\xi|\eta\}'_C$.

□

A similar argument shows that the combinatorial Gromov product is locally constant on the set of regular points:

Lemma 3.6. *Let $\xi, \eta \in \partial\Delta$ be regular points. Then there are neighborhoods $U \ni \xi$, $V \ni \eta$ such that $\{\xi'|\eta'\}_C = \{\xi|\eta\}_C$ for all $\xi' \in B \cap U$, $\eta' \in B \cap V$.*

□

The following result follows easily from Lemmas 3.5, 3.4 and 3.2.

Lemma 3.7. *Let C be a chamber and $\xi_1, \xi_2 \in B$. If $\xi_1\xi_2 \cap$ interior$(C) \neq \phi$, then $\{\xi_1|\xi_2\}_C = \{\xi_1|\xi_2\}'_C = 0$.*

3.3. *Combinatorial cross ratio.* For a regular point $\xi \in B$, we can define the Busemann function B_ξ as follows. Let $\xi \in B$, and C, D be chambers. Let $\{C_i\} \to \xi$ be a geodesic sequence starting from some chamber, and set

$$B_\xi(C, D) = \lim_{i \to +\infty} (|D - C_i| - |C - C_i|).$$

Note that the triangle inequality implies that the limit exists. The arguments in the proofs of Lemma 3.5 and Lemma 3.4 show that $B_\xi(C, D)$ is independent of the choice of $\{C_i\}$ and there is some $i_0 \geq 1$ such that $B_\xi(C, D) = |D - C_i| - |C - C_i|$ for all $i \geq i_0$. It follows that $B_\xi(C, E) = B_\xi(C, D) + B_\xi(D, E)$ for any three chambers C, D, E. It is also easy to see that $B_\xi(C, D)$ is locally constant as a function of regular points ξ: $B_{\xi'}(C, D) = B_\xi(C, D)$ for all regular points ξ' sufficiently close to ξ.

Next we look at how the combinatorial Gromov product changes when the base point changes.

Lemma 3.8. *Let E, E' be two chambers and $\xi, \eta \in B$. Then $\{\xi|\eta\}_{E'} = \{\xi|\eta\}_E + \frac{1}{2}B_\xi(E, E') + \frac{1}{2}B_\eta(E, E')$.*

Proof. Let $\{C_i\}$, $\{D_j\}$ be two geodesic sequences from E to ξ, η respectively, and $\{C'_i\}$, $\{D'_j\}$ be two geodesic sequences from E' to ξ, η respectively. Lemma 3.4 and the first paragraph of this subsection show that there is some $i_0 \geq 1$ such that $\{\xi|\eta\}_E = \{C_i|D_j\}_E$, $\{\xi|\eta\}_{E'} = \{C'_k|D'_l\}_{E'}$, $B_\xi(E, E') = |E' - C_i| - |E - C_i|$ and $B_\eta(E, E') = |E' - D_j| - |E - D_j|$ for all $i, j, k, l \geq i_0$. On the other hand, since ξ and η are regular points, Lemma 3.3 implies that there are $i_1, j_1, k_1, l_1 \geq i_0$ with $C_{i_1} = C'_{k_1}$ and $D_{j_1} = D'_{l_1}$. Now the lemma follows.

□

Lemma 3.8 ensures that the following definition is well-defined.

Definition 3.9. Let $\xi_1, \xi_2, \eta_1, \eta_2 \in B$ be regular points that are pairwise distinct. The combinatorial cross ratio $\{\xi_1\xi_2\eta_1\eta_2\}$ is defined by:

$$\{\xi_1\xi_2\eta_1\eta_2\} = -\{\xi_1|\eta_1\}_C - \{\xi_2|\eta_2\}_C + \{\xi_1|\eta_2\}_C + \{\xi_2|\eta_1\}_C,$$

where C is any chamber of Δ.

Denote $B^4 = \{(\xi_1, \xi_2, \xi_3, \xi_4) : \xi_i \in B,\ \xi_i \neq \xi_j \text{ for } i \neq j\}$. Lemma 3.6 implies that combinatorial cross ratio is locally constant: $\{\xi'_1\xi'_2\eta'_1\eta'_2\} = \{\xi_1\xi_2\eta_1\eta_2\}$ for all $(\xi'_1, \xi'_2, \eta'_1, \eta'_2) \in B^4$ sufficiently close to $(\xi_1, \xi_2, \eta_1, \eta_2) \in B^4$.

We record the following simple lemma for later use.

Lemma 3.10. *Let $e \subset \Delta$ be an edge labeled by i, $x \in$ interior(e) and C_1, C_2 two chambers containing e. Let $\xi \in B$ be a regular point and C the chamber that contains the initial segment of $x\xi$. Then*
(1) $B_\xi(C_1, C_2) = 0$ if $C \neq C_1, C_2$;
(2) $B_\xi(C_1, C_2) = \log q_i$ if $C = C_1$;
(3) $B_\xi(C_1, C_2) = -\log q_i$ if $C = C_2$.

Proof. Let $C' \neq C$ be a chamber containing e. It is clear that $c_{x,\xi} : [0, \infty) \to \Delta$ can be extended to a geodesic $c : (-\epsilon, \infty) \to \Delta$ for some $\epsilon > 0$ such that $c(-\epsilon, 0) \subset \text{interior}(C')$. Now the lemma follows from Lemma 3.2.

□

3.4. *Geometric properties detected by combinatorial cross ratio.* In this section we discuss some geometric properties of a Fuchsian building that can be detected by the combinatorial cross ratio. The results here are crucial to the proof of Theorem 1.3.

Let $\xi, \eta \in \partial\Delta$. The following result says that one can decide whether $\xi\eta$ lies in the 1-skeleton by looking at the behavior of the combinatorial cross ratio.

Lemma 3.11. *Let $\xi, \eta \in \partial\Delta$.*
(1) Suppose $\xi\eta \not\subset \Delta^{(1)}$. Then there are neighborhoods $U_0 \ni \xi$, $V_0 \ni \eta$ in $\partial\Delta$ such that $\{\xi_1\xi_2\eta_1\eta_2\} = 0$ for all $(\xi_1, \xi_2, \eta_1, \eta_2) \in B^4 \cap (U_0 \times U_0 \times V_0 \times V_0)$;
(2) Suppose $\xi\eta \subset \Delta^{(1)}$. Then given any neighborhoods $U_0 \ni \xi$, $V_0 \ni \eta$ in $\partial\Delta$, there exist some integer i, $1 \le i \le k$, open subsets $U, V \subset \partial\Delta$ and open subsets $W_1, W_2 \subset U \times U \times V \times V$ with the following properties: $\xi \in U \subset U_0$, $\eta \in V \subset V_0$; $\{\xi_1\xi_2\eta_1\eta_2\}$ is an integral multiple of $\frac{1}{2}\log q_i$ for all $(\xi_1, \xi_2, \eta_1, \eta_2) \in B^4 \cap (U \times U \times V \times V)$; $\{\xi_1\xi_2\eta_1\eta_2\} = -\frac{1}{2}\log q_i$ for all $(\xi_1, \xi_2, \eta_1, \eta_2) \in B^4 \cap W_1$, and $\{\xi_1\xi_2\eta_1\eta_2\} = 0$ for all $(\xi_1, \xi_2, \eta_1, \eta_2) \in B^4 \cap W_2$.

Proof. (1). Suppose $\xi\eta$ is not contained in $\Delta^{(1)}$. Then $\xi\eta$ intersects the interior of some chamber C. There are neighborhoods $U_0 \ni \xi$ and $V_0 \ni \eta$ such that $\xi'\eta'$ meets the interior of C for all $\xi' \in U_0$, $\eta' \in V_0$. Then Lemma 3.7 implies that $\{\xi'|\eta'\}_C = 0$ for all $\xi' \in B \cap U_0$, $\eta' \in B \cap V_0$. Now (1) follows.

(2). Now suppose $\xi\eta$ is contained in $\Delta^{(1)}$. Let $e \subset \xi\eta$ be an edge, which is labeled by some i. Given any neighborhoods $U_0 \ni \xi$, $V_0 \ni \eta$ in $\partial\Delta$, we choose open subsets $U, V \subset \partial\Delta$ with $\xi \in U \subset U_0$, $\eta \in V \subset V_0$ such that for any $\xi' \in U$ and $\eta' \in V$, the geodesic $\xi'\eta'$ either contains e or intersects the interior of some chamber containing e. Let C be a fixed chamber containing e, and $\xi' \in B \cap U$, $\eta' \in B \cap V$. Then there is some chamber C' containing e with $\xi'\eta' \cap \text{interior}(C') \ne \phi$. Lemma 3.7 implies that $\{\xi'|\eta'\}_{C'} = 0$. On the other hand, Lemma 3.10 shows that $B_{\xi'}(C', C)$ and $B_{\eta'}(C', C)$ are integral multiples of $\log q_i$. It follows from Lemma 3.8 that $\{\xi'|\eta'\}_C$ is an integral multiple of $\frac{1}{2}\log q_i$. Consequently $\{\xi_1\xi_2\eta_1\eta_2\}$ is an integral multiple of $\frac{1}{2}\log q_i$ for all $(\xi_1, \xi_2, \eta_1, \eta_2) \in B^4 \cap (U \times U \times V \times V)$.

We claim there exist $(\xi_1, \xi_2, \eta_1, \eta_2), (\bar{\xi_1}, \bar{\xi_2}, \bar{\eta_1}, \bar{\eta_2}) \in B^4 \cap (U \times U \times V \times V)$ such that $\{\xi_1\xi_2\eta_1\eta_2\} = -\frac{1}{2}\log q_i$ and $\{\bar{\xi_1}\bar{\xi_2}\bar{\eta_1}\bar{\eta_2}\} = 0$. Since the combinatorial cross ratio is locally constant, there are open subsets $W_1, W_2 \subset U \times U \times V \times V$ with $(\xi_1, \xi_2, \eta_1, \eta_2) \in W_1$, $(\bar{\xi_1}, \bar{\xi_2}, \bar{\eta_1}, \bar{\eta_2}) \in W_2$ such that the combinatorial cross ratio is $-\frac{1}{2}\log q_i$ on $B^4 \cap W_1$ and is 0 on $B^4 \cap W_2$. We next prove the claim.

Let A be an apartment containing $\xi\eta$. Denote by C_1, C_2 the two chambers in A containing e, and $C_3 \ne C_1, C_2$ some other chamber containing e. Let m be the midpoint of e. We choose $\xi_1' \in U \cap \partial A$, $\eta_1', \eta_2' \in V \cap \partial A$ and $\xi_2' \in U$ with the following properties:
(a) C_1 contains the initial segments of $m\xi_1'$ and $m\eta_1'$, C_2 contains the initial segment of $m\eta_2'$, and C_3 contains the initial segment of $m\xi_2'$;
(b) $\angle_m(\xi_i', \xi) = \angle_m(\eta_j', \eta)$ for all $i, j = 1, 2$;
(c) $\xi_1'\eta_1' \cap \text{interior}(C_1) \ne \phi$.
Notice (b) implies that $m \in \xi_1'\eta_2', \xi_2'\eta_1', \xi_2'\eta_2'$. We can choose regular points $\xi_1, \xi_2 \in U$, $\eta_1, \eta_2 \in V$ sufficiently close to $\xi_1', \xi_2', \eta_1', \eta_2'$ respectively such that the following conditions are satisfied:
(I) $\xi_1\eta_j \cap \text{interior}(C_1) \ne \phi$ $(j = 1, 2)$, $\xi_2\eta_1 \cap \text{interior}(C_1) \ne \phi$;
(II) $\xi_2\eta_2 \cap \text{interior}(C_k) \ne \phi$ for $k = 2, 3$.
By Lemma 3.7 $\{\xi_1|\eta_j\}_{C_1} = \{\xi_2|\eta_1\}_{C_1} = \{\xi_2|\eta_2\}_{C_2} = 0$. Notice (II) implies that $\xi_2\eta_2 \cap \text{interior}(e) \ne \phi$ and $\xi_2\eta_2 \cap \text{interior}(C_1) = \phi$ since $C_2 \cup C_3$ is convex in Δ and e separates C_2 from C_3. It follows from Lemma 3.10 that $B_{\xi_2}(C_2, C_1) = 0$ and $B_{\eta_2}(C_2, C_1) = \log q_i$. Now Lemma 3.8 implies that $\{\xi_2|\eta_2\}_{C_1} = \frac{1}{2}\log q_i$. Consequently $\{\xi_1\xi_2\eta_1\eta_2\} = -\frac{1}{2}\log q_i$. Finally if one chooses $\bar{\xi_1}, \bar{\xi_2} \in U$ sufficiently close to ξ_1', and also chooses $\bar{\eta_1}, \bar{\eta_2} \in V$ sufficiently close to η_1', then we have $\{\bar{\xi_1}\bar{\xi_2}\bar{\eta_1}\bar{\eta_2}\} = 0$. The proof is now complete.

□

Let $\eta \in \partial\Delta$ be a singular point. Then η is represented by a ray $\sigma : [0, \infty) \to \Delta$ contained in $\Delta^{(1)}$. Let $\xi_1, \xi_2 \in \Delta \cup \partial\Delta - \{\xi\}$ be such that $\xi_1\eta \cap \sigma = \xi_2\eta \cap \sigma = \phi$. We parameterize $\xi_i\eta$ $(i = 1, 2)$ by $\sigma_i : I_i \to \Delta$ such that $\sigma_i(t)$ and $\sigma(t)$ lie on the same horosphere centered at η, where $I_i \subset \mathbb{R}$ is an interval. Since Δ is CAT(-1), we have $d(\sigma_i(t), \sigma(t)) \to 0$ as $t \to \infty$. Fix a very large t_0 such that $\sigma(t_0)$ is the midpoint of some edge $e \subset \sigma$. Then $\sigma_i(t_0)$ must lie in the interior of some chamber C_i containing e. We say ξ_1 *and* ξ_2 *lie at different sides of* η if $C_1 \ne C_2$, and say ξ_1 *and* ξ_2 *lie on the same side of* η if $C_1 = C_2$. It is not hard to see that this definition is well-defined.

The following result says that, for a singular point η represented by a ray $\sigma : [0,\infty) \to \Delta$ contained in $\Delta^{(1)}$, and $\xi_1, \xi_2 \in \partial\Delta$ with $\xi_1\eta \cap \sigma = \xi_2\eta \cap \sigma = \phi$, whether ξ_1, ξ_2 lie at different sides of η can be detected by the combinatorial cross ratio.

Lemma 3.12. *Let $\eta \in \partial\Delta$ be a singular point represented by a ray $\sigma : [0,\infty) \to \Delta$ contained in $\Delta^{(1)}$, and $\xi_1, \xi_2 \in \partial\Delta$ with $\xi_1\eta \cap \sigma = \xi_2\eta \cap \sigma = \phi$.*
(1) If ξ_1 and ξ_2 lie on the same side of η, then there are pairwise disjoint open subsets $U_1, U_2, V_1, V_2 \subset \partial\Delta$ that satisfy the following conditions: $\xi_1 \in U_1$, $\xi_2 \in U_2$, $\eta \in V_2$; the combinatorial cross ratio is constant on $B^4 \cap (U_1 \times U_2 \times V_1 \times V_2)$.
(2) If ξ_1 and ξ_2 lie on different sides of η, then for any pairwise disjoint open subsets $U_1, U_2, V_1, V_2 \subset \partial\Delta$ satisfying $\xi_1 \in U_1$, $\xi_2 \in U_2$, $\eta \in V_2$, there are open subsets $W_1, W_2 \subset U_1 \times U_2 \times V_1 \times V_2$ and constants $c_1 \neq c_2$ such that: $\{\xi_1'\xi_2'\eta_1'\eta_2'\} = c_1$ for all $(\xi_1', \xi_2', \eta_1', \eta_2') \in B^4 \cap W_1$ and $\{\xi_1'\xi_2'\eta_1'\eta_2'\} = c_2$ for all $(\xi_1', \xi_2', \eta_1', \eta_2') \in B^4 \cap W_2$.

Proof. (1). Suppose ξ_1, ξ_2 lie at the same side of η. Then there is a chamber C that contains an edge $e \subset \sigma$ such that both $\xi_1\eta$ and $\xi_2\eta$ intersect the interior of C. Then there are pairwise disjoint neighborhoods $U_i \ni \xi_i (i = 1, 2)$, $V \ni \eta$ such that for any $\xi_1' \in B \cap U_1$, $\xi_2' \in B \cap U_2$ and $\eta' \in B \cap V$, both $\xi_1'\eta'$ and $\xi_2'\eta'$ intersect the interior of C. Lemma 3.7 implies that $\{\xi_1'|\eta'\}_C = \{\xi_2'|\eta'\}_C = 0$. Now choose disjoint open subsets $V_1, V_2 \subset V$ such that $\eta \in V_2$. Then it follows from the definition that $\{\xi_1'\xi_2'\eta_1'\eta_2'\} = 0$ for all $(\xi_1', \xi_2', \eta_1', \eta_2') \in B^4 \cap (U_1 \times U_2 \times V_1 \times V_2)$.

(2). Now suppose ξ_1, ξ_2 lie at different sides of η. We claim for any pairwise disjoint open subsets $U_1, U_2, V_1, V_2 \subset \partial\Delta$ satisfying $\xi_1 \in U_1$, $\xi_2 \in U_2$, $\eta \in V_2$, there exist $\xi_1' \in B \cap U_1, \xi_2' \in B \cap U_2, \eta_1' \in B \cap V_1$ and $\eta_2', \eta_2'' \in B \cap V_2$ such that $\{\xi_1'\xi_2'\eta_1'\eta_2'\} \neq \{\xi_1'\xi_2'\eta_1'\eta_2''\}$. Let $c_1 = \{\xi_1'\xi_2'\eta_1'\eta_2'\}$, $c_2 = \{\xi_1'\xi_2'\eta_1'\eta_2''\}$. Since the combinatorial cross ratio is locally constant, there are open subsets $W_1, W_2 \subset U_1 \times U_2 \times V_1 \times V_2$ satisfying $(\xi_1', \xi_2', \eta_1', \eta_2') \in W_1$, $(\xi_1', \xi_2', \eta_1', \eta_2'') \in W_2$ such that the combinatorial cross ratio takes the constant value c_i $(i = 1, 2)$ on $B^4 \cap W_i$. Next we prove the claim.

Since ξ_1, ξ_2 lie at different sides of η, there are chambers $C_1 \neq C_2$ both containing an edge $e \subset \sigma$ such that $\xi_i\eta \cap \text{interior}(C_i) \neq \phi$ $(i = 1, 2)$. By shrinking the neighborhoods V_2, U_1, U_2 we may assume $\xi_i'\eta' \cap \text{interior}(C_i) \neq \phi$ $(i = 1, 2)$ for all $\eta' \in B \cap V_2$, $\xi_i' \in B \cap U_i$. Lemma 3.7 implies that $\{\xi_i'|\eta'\}_{C_i} = 0$. We fix some $\xi_i' \in B \cap U_i (i = 1, 2)$ and $\eta_1' \in V_1$. By the definition of combinatorial cross ratio we only need to find $\eta_2', \eta_2'' \in B \cap V_2$ such that $\{\xi_2'|\eta_2'\}_{C_1} \neq \{\xi_2'|\eta_2''\}_{C_1}$. By Lemma 3.8, $\{\xi_2'|\eta'\}_{C_1} = \{\xi_2'|\eta'\}_{C_2} + \frac{1}{2}B_{\xi_2'}(C_2, C_1) + \frac{1}{2}B_{\eta'}(C_2, C_1)$. $B_{\xi_2'}(C_2, C_1)$ is independent of η' and we have observed that $\{\xi_2'|\eta'\}_{C_2} = 0$. Let m be the midpoint of e. For $\eta' \in B \cap V_2$, the initial segment of $m\eta'$ could lie in C_1, C_2 or some other chamber C_3 containing e. Lemma 3.10 shows that the values for $B_{\eta'}(C_2, C_1)$ are different in these three cases. Consequently the values of $\{\xi_1'\xi_2'\eta_1'\eta'\}$ are also different, and the proof is complete.

□

4. **Sullivan's approach and Bourdon's work on Fuchsian buildings.** In this section we review Bourdon's work on Fuchsian buildings and Sullivan's approach to Mostow rigidity.

4.1. *Combinatorial metrics on the boundary.* Combinatorial metrics were introduced and studied by M. Bourdon ([Bo1], [Bo2]). Here we recall the definition and some facts about combinatorial metrics. Let A_R be the labeled 2-complex of Section 2.1 and $\mathcal{G}_{A_R}$ its dual graph. An edge of $\mathcal{G}_{A_R}$ has length $\log q_i$ if the corresponding chambers share an edge labeled by i. Let $|\cdot|$ be the induced path metric. For any integer $n \geq 1$, let $a(n)$ be the number of vertices that are at distance at most n from a fixed vertex C of $\mathcal{G}_{A_R}$. The growth rate of $\mathcal{G}_{A_R}$ is defined by:

$$\tau = \limsup_{n\to\infty} \frac{1}{n} \log a(n).$$

Let C be a chamber of Δ. For $\xi \in \partial\Delta$ and $r > 0$, we introduce the following subset of $\partial\Delta$:

$$B_C(\xi, r) = \{\eta \in \partial\Delta : e^{-\tau\{\xi|\eta\}_C} \leq r\}.$$

For a continuous path γ in $\partial\Delta$, we define

$$l(\gamma) = \lim_{r\to 0} \inf\{\Sigma_i r_i\},$$

where the infimum is taken over all finite coverings $\{B_C(\xi_i, r_i)\}$ of γ, with $\xi_i \in \gamma$ and $r_i \leq r$. Finally, for $\xi, \eta \in \partial\Delta$ we set

$$\delta_C(\xi, \eta) = \inf l(\gamma),$$

where γ varies over all continuous path in $\partial\Delta$ from ξ to η.

It is shown by M. Bourdon ([Bo1], 3.1.4) that δ_C defines a metric on $\partial\Delta$ and has the following property: there is a constant $\lambda \geq 1$ such that

$$\frac{1}{\lambda} e^{-\tau\{\xi|\eta\}_C} \leq \delta_C(\xi, \eta) \leq \lambda e^{-\tau\{\xi|\eta\}_C}$$

for all $\xi, \eta \in \partial\Delta$.

We shall show that for any two chambers C, D, the combinatorial metrics δ_C and δ_D are conformally equivalent. We first recall the definition of conformal maps.

Let $f : X \to Y$ be a homeomorphism between metric spaces. The map f is *quasi-symmetric* if there exists a homeomorphism $\phi : [0, \infty) \to [0, \infty)$ so that

$$d_X(x, a) \leq t d_X(x, b) \Rightarrow d_Y(f(x), f(a)) \leq \phi(t) d_Y(f(x), f(b))$$

for all $x, a, b \in X$ and all $t \in [0, \infty)$.

Let $f : X \to Y$ be a homeomorphism between metric spaces. For any $x \in X$ and $r > 0$, let

$$L_f(x, r) = \sup\{d_Y(f(x), f(x')) : d_X(x, x') \leq r\},$$
$$l_f(x, r) = \inf\{d_Y(f(x), f(x')) : d_X(x, x') \geq r\},$$
$$L_f(x) = \limsup_{r\to 0} \frac{L_f(x, r)}{r},$$
$$l_f(x) = \liminf_{r\to 0} \frac{l_f(x, r)}{r}.$$

Assume X and Y have finite Hausdorff dimensions. Denote by H_X and H_Y their Hausdorff dimensions and by $\mathcal{H}_X$ and $\mathcal{H}_Y$ their Hausdorff measures (see [F] for definitions). We say that f is *conformal* if f is quasi-symmetric and satisfies
(i) $L_f(x) = l_f(x) \in (0, \infty)$ for $\mathcal{H}_X$-almost every $x \in X$;
(ii) $L_{f^{-1}}(y) = l_{f^{-1}}(y) \in (0, \infty)$ for $\mathcal{H}_Y$-almost every $y \in Y$.

Recall (see the proof of Lemma 2.2.7 in [Bo2]) that for any chamber C, the set B has full $\mathcal{H}_{\delta_C}$-measure in $\partial\Delta$.

Lemma 4.1. *Let C, D be two chambers of Δ. Then δ_C and δ_D are conformally equivalent, that is, the identity map $id : (\partial\Delta, \delta_C) \to (\partial\Delta, \delta_D)$ is a conformal map. Furthermore, for any regular point ξ, there is a neighborhood V of ξ in $\partial\Delta$ such that $\delta_D(\xi, \eta) = e^{-\tau B_\xi(C,D)} \delta_C(\xi, \eta)$ for all $\eta \in V$.*

Proof. Triangle inequality implies $-|C-D| \leq \{\xi|\eta\}_C - \{\xi|\eta\}_D \leq |C-D|$. Since for any chamber C', the combinatorial distance $\delta_{C'}(\xi, \eta)$ is comparable with $e^{-\tau\{\xi|\eta\}_{C'}}$, we see id : $(\partial\Delta, \delta_C) \to (\partial\Delta, \delta_D)$ is bi-Lipschitz, in particular, it is quasi-symmetric.

We first assume C and D share an edge e. Then e is labeled by some i. Denote by m the midpoint of e. Let $\xi \in \partial\Delta$ be a regular point. The initial segment of $m\xi$ lies in C, D or some chamber $E \neq C, D$ containing e. Suppose the initial segment of $m\xi$ lies in C. Then there is a neighborhood $U \subset \Delta \cup \partial\Delta$ of ξ such that the initial segment of $m\eta$ lies in C for all $\eta \in U$. All these geodesic rays (or segments) can be extended into D. It follows that $|D - C'| = \log q_i + |C - C'|$ for all chambers $C' \subset U$. Now it is clear that $\{\eta_1|\eta_2\}_D = \log q_i + \{\eta_1|\eta_2\}_C$ for all $\eta_1, \eta_2 \in U \cap \partial\Delta$. It implies that for $\xi' \in U \cap \partial\Delta$ and sufficiently small $r > 0$ we have $B_D(\xi', r) = B_C(\xi', re^{\tau \log q_i})$. The definition of δ_C now implies that $\delta_D(\xi, \eta) = e^{-\tau \log q_i} \delta_C(\xi, \eta)$ for all $\eta \in U \cap \partial\Delta$. Note $\log q_i = B_\xi(C, D)$. Hence $\delta_D(\xi, \eta) = e^{-\tau B_\xi(C,D)} \delta_C(\xi, \eta)$ for all $\eta \in U \cap \partial\Delta$, which shows $L_f(\xi) = l_f(\xi) = e^{-\tau B_\xi(C,D)}$. Here

$f = \mathrm{id} : (\partial\Delta, \delta_C) \to (\partial\Delta, \delta_D)$. Similarly one arrives at the same conclusion when the initial segment of $m\xi$ lies in D or some $E \neq C, D$. Since the regular set B has full $\mathcal{H}_{\delta_C}$-measure in $\partial\Delta$, $\mathrm{id} : (\partial\Delta, \delta_C) \to (\partial\Delta, \delta_D)$ is a conformal map.

Now let C, D be two arbitrary chambers. Then δ_C and δ_D are still conformally equivalent since there exists a gallery from C to D. Since $B_\xi(C_1, C_3) = B_\xi(C_1, C_2) + B_\xi(C_2, C_3)$ holds for any $\xi \in B$ and any three chambers C_1, C_2, C_3, the formula $\delta_D(\xi, \eta) = e^{-\tau B_\xi(C,D)}\delta_C(\xi, \eta)$ holds for all η sufficiently close to ξ.

□

If $g : \Delta \to \Delta$ is an isometry, then it is clear that the induced map on the boundary $g : (\partial\Delta, \delta_C) \to (\partial\Delta, \delta_{g(C)})$ is an isometry for any chamber C. By Lemma 4.1, $g : (\partial\Delta, \delta_C) \to (\partial\Delta, \delta_C)$ is a conformal map and $\delta_C(g(\xi), g(\eta)) = e^{-\tau B_\xi(C, g^{-1}(C))}\delta_C(\xi, \eta)$ for any fixed regular point ξ and all η in a small neighborhood of ξ.

Since δ_C and δ_D are bi-Lipschitz, they have the same Hausdorff dimension. Let H be their common Hausdorff dimension. Let $\mathcal{H}_C$ be the Hausdorff measure of δ_C. Lemma 4.1 implies that $\mathcal{H}_C$ and $\mathcal{H}_D$ are in the same measure class and $\mathcal{H}_D(\xi) = e^{-H\tau B_\xi(C,D)}\mathcal{H}_C(\xi)$ for all regular points ξ. If $g : \Delta \to \Delta$ is an isometry, then we have $g^*\mathcal{H}_C(\xi) = e^{-H\tau B_\xi(C, g^{-1}(C))}\mathcal{H}_C(\xi)$. Hence the measures $\{\mathcal{H}_C\}$ are the so-called conformal measures with respect to $Isom(\Delta)$.

4.2. *Sullivan's approach.* Let Δ_1, Δ_2 be two Fuchsian buildings, and G a group acting properly and cocompactly on both Δ_1 and Δ_2. Then there is an equivariant homeomorphism $h : \partial\Delta_1 \to \partial\Delta_2$. There are two steps in Sullivan's approach. The first step is to show h preserves the combinatorial cross ratio almost everywhere (with respect to certain measures). The second step is to show the implication: "h preserves the combinatorial cross ratio almost everywhere" ⇒ "h extends to an isomorphism from Δ_1 to Δ_2".

The measures appearing in Step 1 are the so-called conformal measures. In our case they are Hausdorff measures $\{\mathcal{H}_C\}$ of the combinatorial metrics, as introduced in Section 4.1. Sullivan's ergodic arguments (see [S2]) show that if the Hausdorff dimensions of the combinatorial metrics on $\partial\Delta_1$ and $\partial\Delta_2$ are the same and h is nonsingular with respect to the Hausdorff measures, then h preserves the combinatorial cross ratio almost everywhere with respect to the Hausdorff measures. Here "h is nonsingular" means a subset $X \subset \partial\Delta_1$ has measure 0 if and only if $h(X)$ has measure 0. Hence to complete Step 1, one has to show
(1) the Hausdorff dimensions of the combinatorial metrics on $\partial\Delta_1$ and $\partial\Delta_2$ are the same;
(2) h is nonsingular with respect to the Hausdorff measures.
These two facts have been verified by M. Bourdon (see [Bo1], [Bo2]), and Step 1 is established. Step 2 follows from Theorem 1.3.

Remark 4.2. As mentioned at the beginning of Section 3 our definitions of combinatorial cross ratio and combinatorial metric are slightly different from those of M. Bourdon since we only used vertices in the dual graph while M. Bourdon used the whole dual graph. Since the vertices form a net in the dual graph, our combinatorial cross ratio differs from M. Bourdon's by an additive constant that depend only on the building. It follows that the two combinatorial metrics are bi-Lipschitz equivalent with the Lipschitz constant depending only on the building, and the corresponding Hausdorff measures are in the same measure class.

5. **Triangles and quadrilaterals in the 1-skeleton.** In this section we state the main results concerning triangles and quadrilaterals contained in the 1-skeleton of Δ. Their proofs are somehow tedious and are contained in Section 10.

Let Δ be a Fuchsian building. A subset $T \subset \Delta$ is called a *triangle* if there are points $x, y, z \in \Delta$ such that $T = xy \cup yz \cup zx$. In this case, we also use the notation $T = (x, y, z)$. The three points x, y, z shall be called the corners of T, and xy, yz, zx called the sides of T. Similarly, we say a subset $Q \subset \Delta$ is a *quadrilateral* if there are $x_1, x_2, x_3, x_4 \in \Delta$ such that $Q = x_1x_2 \cup x_2x_3 \cup x_3x_4 \cup x_4x_1$.

We use the notation $Q = (x_1, x_2, x_3, x_4)$, and call x_i a corner of Q and $x_i x_{i+1}$ a side of Q. We are mainly interested in triangles and quadrilaterals that are contained in the 1-skeleton of Δ.

Let $D \subset \Delta$ be a finite subcomplex that is homeomorphic to a compact surface with boundary. A vertex $v \in D$ is a *special point* if $v \in \text{interior}(D)$ and $\text{length}(\text{Link}(D, v)) > 2\pi$, or $v \in \partial D$ and $\text{length}(\text{Link}(D, v)) > \pi$. The following result implies that each triangle in $\Delta^{(1)}$ bounds a convex disk.

Proposition 5.1. *Let Δ be a Fuchsian building and $T \subset \Delta^{(1)}$ a triangle that is homeomorphic to a circle. Then there is a finite subcomplex $S(T)$ with the following properties:*
(1) $S(T)$ is homeomorphic to a closed disk with boundary T;
(2) $S(T)$ has no special points.
In particular, $S(T)$ is convex in Δ.

We list all the triangles in $\Delta^{(1)}$. Recall for integers $m_1, m_2, m_3 \geq 2$, (m_1, m_2, m_3) denotes the triangle (unique up to isometry) in $\mathbb{H}^2$ with angles π/m_1, π/m_2 and π/m_3.

Proposition 5.2. *Let Δ be a Fuchsian building with chamber R, and $T \subset \Delta^{(1)}$ a triangle homeomorphic to a circle.*
(1) If R is not a triangle, then there is no triangle in $\Delta^{(1)}$ homeomorphic to a circle;
(2) If R is a triangle with all angles $< \pi/2$, then $S(T)$ is a chamber;
(3) If $R = (2, m_1, m_2)$ with $(m_1, m_2) = (6, 6), (6, 8)$ or $(8, 8)$, then $S(T)$ is isomorphic to one of the labeled complexes in Figure 1;

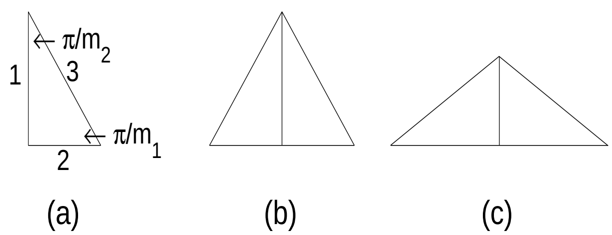

FIGURE 1. $S(T)$ when $R = (2, m_1, m_2)$ with $(m_1, m_2) = (6, 6), (6, 8)$ or $(8, 8)$

(4) If $R = (2, 4, 6)$ or $(2, 4, 8)$, then $S(T)$ is isomorphic to one of the labeled complexes in Figure 2;

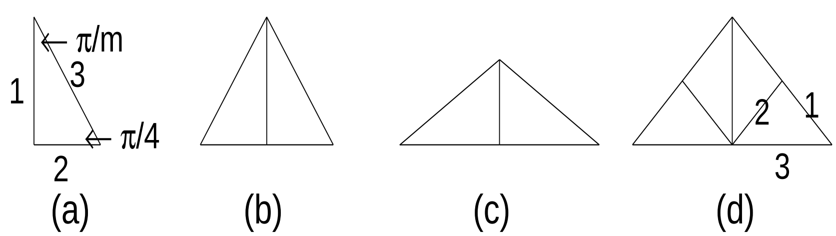

FIGURE 2. $S(T)$ when $R = (2, 4, m)$ with $m = 6$ or 8

(5) If $R = (2, 3, 8)$, then $S(T)$ is isomorphic to one of the labeled complexes in Figure 3.

We let A_0 denote the area of a chamber. For any quadrilateral $Q = (x_1, x_2, x_3, x_4)$, let $\Sigma(Q) = \Sigma_{i=1}^{4} \angle_{x_i}(x_{i-1}, x_{i+1})$ be the sum of angles at the 4 corners of Q.

Proposition 5.3. *Let Δ be a Fuchsian building, and $Q \subset \Delta^{(1)}$ a quadrilateral homeomorphic to a circle. Then there is a finite subcomplex $S(Q)$ with the following properties:*
(1) $S(Q)$ is homeomorphic to a closed disk with boundary Q;
(2) $2\pi \geq \Sigma(Q) + n(Q)A_0$, where $n(Q)$ is the number of chambers in $S(Q)$.

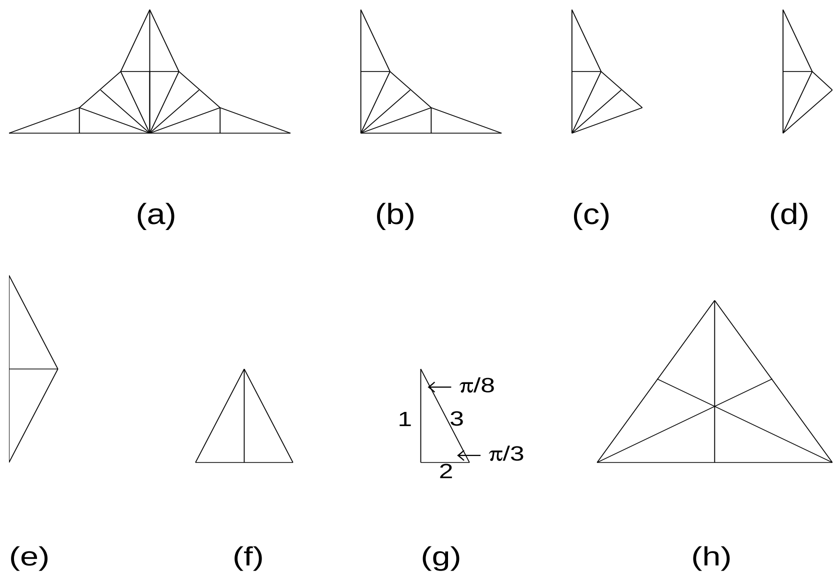

FIGURE 3. $S(T)$ when $R = (2, 3, 8)$

Recall the chamber R is a compact convex k-gon in $\mathbb{H}^2$. We let α_0 be the smallest angle of R. Note $A_0 \geq \pi/2$ if $k \geq 5$. There are only 6 right triangles that can occur as the chambers of Fuchsian buildings: $(2, 8, 8)$, $(2, 6, 6)$, $(2, 6, 8)$, $(2, 4, 6)$, $(2, 4, 8)$, $(2, 3, 8)$. Their areas are respectively: $\pi/4$, $\pi/6$, $5\pi/24$, $\pi/12$, $\pi/8$, $\pi/24$.

Corollary 5.4. *Let Δ be a Fuchsian building whose chamber R is a k-gon with $k \geq 5$, and $Q \subset \Delta^{(1)}$ a quadrilateral. Then Q is not homeomorphic to a circle.*

Proof. Suppose Q is homeomorphic to a circle. Since R is a k-gon with $k \geq 5$ and all its angles are $\leq \pi/2$, we have $A_0 \geq \pi - \alpha_0$. Proposition 5.3 implies $2\pi \geq 4\alpha_0 + n(Q)(\pi - \alpha_0)$. It is easy to verify that $n(Q) \leq 1$ for each of the cases $\alpha_0 = \pi/2, \pi/3, \pi/4, \pi/6, \pi/8$, which means Q is the boundary of a chamber. This is a contradiction since Q is a quadrilateral and R is a k-gon with $k \geq 5$.

□

Recall for any vertex $v \in \Delta$, the link $\mathrm{Link}(\Delta, v)$ is a generalized m-gon for some $m \geq 2$. We set $m(v) = m$ if $\mathrm{Link}(\Delta, v)$ is a generalized m-gon and say v is indexed by $m(v)$. The vertices in $\mathrm{Link}(\Delta, v)$ are divided into two types, and two vertices are of the same type if and only if the distance between them is an even multiple of π/m. Let $vx, vy \subset \Delta^{(1)}$ be two (nondegenerate) geodesics starting from v. The angle $\angle_v(x, y)$ is called *even* if it is an even multiple of π/m, and is called *odd* if it is an odd multiple of π/m. Equivalently, $\angle_v(x, y)$ is even if and only if the initial directions of vx and vy at v have the same type in $\mathrm{Link}(\Delta, v)$.

Corollary 5.5. *Let Δ be a Fuchsian building with chamber R, and $Q = (x, y, z, w) \subset \Delta^{(1)}$ a quadrilateral homeomorphic to a circle. Suppose Q has even angles at x and y, and xy contains at least two vertices in the interior. Then $R = (2, 3, 8)$.*

Proof. Let $v_1 \neq v_2 \in \mathrm{interior}(xy)$ be two vertices with $v_1 \in \mathrm{interior}(xv_2)$. We count the chambers in $S(Q)$ that intersect xy. There are at least $m(v_1)$ chambers (in $S(Q)$) incident to v_1, at least $m(v_2)$ incident to v_2; but there might be one chamber incident to both v_1 and v_2. Since Q has an even angle at x, there are at least 2 chambers incident to x, but one of them could also be incident to v_1. Similarly for y. Summarize, there are at least $m(v_1) + m(v_2) + 1$ chambers in

$S(Q)$, that is, $n(Q) \geq m(v_1) + m(v_2) + 1$. In particular, $n(Q) \geq 5$ always holds. Also note $\Sigma(Q) \geq \frac{\pi}{m(z)} + \frac{\pi}{m(w)} + \frac{2\pi}{m(x)} + \frac{2\pi}{m(y)}$. In particular, $\Sigma(Q) \geq 3\pi/4$ always holds.

We suppose $R \neq (2,3,8)$ and will derive a contradiction from this. We consider several cases.

Case 1: $k \geq 5$. In this case $A_0 \geq \pi/2$. Proposition 5.3 implies $n(Q) \leq 2$, contradicting to the fact $n(Q) \geq 5$.

Case 2: $k = 4$. If $\alpha_0 = \pi/8$, then $A_0 \geq 3\pi/8$. Proposition 5.3 implies $n(Q) \leq 3$, contradiction. If $\alpha_0 = \pi/6$, then $A_0 \geq \pi/3$ and $\Sigma(Q) \geq \pi$. Proposition 5.3 again implies $n(Q) \leq 3$. We similarly obtain contradictions when $\alpha_0 = \pi/4$ or $\alpha_0 = \pi/3$.

Below we will apply Proposition 5.3 without mentioning it.

Case 3: R is a triangle with no right angle. In this case $m(v_1), m(v_2) \geq 3$ and we have $n(Q) \geq 7$. If $\alpha_0 = \pi/8$, then $A_0 \geq 5\pi/24$ and $n(Q) \leq 6$, contradiction. Similarly one obtains contradictions when $\alpha_0 = \pi/6$ or $\alpha_0 = \pi/4$.

Case 4: $R \neq (2,3,8)$ is a right triangle. We only give the details for $R = (2,4,6)$, the proofs for the other 4 cases are similar. So assume $R = (2,4,6)$. In this case, $A_0 = \pi/12$. Consider any complete geodesic c containing xy. The vertices on c are alternatingly indexed by m_1 and m_2, where $(m_1, m_2) = (2,6), (4,6)$ or $(2,4)$. Notice none of x, y is indexed by 2, otherwise the quadrilateral is actually a triangle and one of its sides contains at least 4 edges, contradicting to Proposition 5.2. First assume $(m_1, m_2) = (2,4)$. Then $m(x) = m(y) = 4$ and $n(Q) \leq 8$. There is at least one vertex $v \in \text{interior}(xy)$ indexed by 4. Also recall the angles of Q at x and y are even. It follows that there are at least 4 chambers in $S(Q)$ incident to v and at least two chambers incident to each of x, y, for a total of 8. Hence $S(Q)$ is exactly the union of these 8 chambers. But this union is a hexagon, not a quadrilateral.

Next assume $(m_1, m_2) = (4,6)$. Suppose $\{m(x), m(y)\} = \{4,6\}$. Then $n(Q) \leq 10$. There are two vertices $v_1, v_2 \in \text{interior}(xy)$ with $m(v_1) = 6$, $m(v_2) = 4$. It follows that $n(Q) \geq 6+4+1 = 11$, contradiction. One similarly obtains contradictions when $m(x) = m(y) = 4$ or 6.

Finally we assume $(m_1, m_2) = (2,6)$. Then $m(x) = m(y) = 6$ and $n(Q) \leq 12$. There is at least one vertex $v \in \text{interior}(xy)$ with $m(v) = 6$. We count chambers in $S(Q)$ incident to vertices indexed by 6: (at least) 6 chambers incident to v, 2 incident to each of x, y. So $n(Q) \geq 10$. Proposition 5.3 implies the angles at x and y are $\pi/3$ and none of the other two angles of Q is $\pi/2$. It follows that the vertices on xw are alternatingly indexed by 2 and 6, and $m(w) \neq 2$. Hence $m(w) = 6$. Similarly $m(z) = 6$. Notice interior(xw) contains exactly one vertex and so does interior(yz). There is (at least) one chamber in $S(Q)$ incident to each of z, w. We have exhibited 12 chambers in $S(Q)$. It follows that $S(Q)$ is the union of these 12 chambers. However, one can check that this union is not a quadrilateral.

□

Corollary 5.6. *Let Δ be a Fuchsian building with chamber R, and $Q = (x,y,z,w) \subset \Delta^{(1)}$ a quadrilateral homeomorphic to a circle. If Q has three even angles, then R is a right triangle.*

Proof. Suppose R is not a right triangle. We may assume the angles at x, y and z are even. We count the chambers in $S(Q)$ that intersect xy: at least 2 chambers incident to x and y each, but there might be a chamber incident to both x and y. Hence $n(Q) \geq 3$. On the other hand, the assumption implies $\Sigma(Q) \geq 7\alpha_0$, where α_0 is the smallest angle of R. Recall R is a k-gon.

Case 1: $k \geq 5$. In this case, $A_0 \geq \pi/2$ and $\Sigma(Q) \geq 7\pi/8$. Proposition 5.3 (2) implies $n(Q) \leq 2$, contradicting to the fact $n(Q) \geq 3$.

Case 2: $k = 4$. As above, Proposition 5.3 (2) implies $n(Q) \leq 2$ if $\alpha_0 = \pi/6$, $\pi/4$ or $\pi/3$. Assume $\alpha_0 = \pi/8$. Then $n(Q) \leq 3$. It follows that $S(Q)$ is the union of the three chambers indicated in the first paragraph. However, this union is not a quadrilateral.

Case 3: R is a triangle with no right angle. Then Proposition 5.3 (2) implies $n(Q) \leq 5$ in each of the cases: $\alpha_0 = \pi/8, \pi/6$ or $\pi/4$. Let $\alpha_0 \leq \beta_0 \leq \gamma_0 \leq \pi/3$ be the three angles of R. We first

assume one of the segments xy, yz contains a vertex v in the interior, say $v \in \text{interior}(xy)$. Then $m(v) \geq 3$ and there are at least 3 chambers in $S(Q)$ incident to v. On the other hand, there are at least two chambers in $S(Q)$ incident to x, and at least one of them is distinct from the three chambers incident to v. Similarly there is at least one chamber in $S(Q)$ incident to y and different from the three chambers incident to v. Combining with the observation $n(Q) \leq 5$, we see $n(Q) = 5$. However, the union of these 5 chambers is either a pentagon or a hexagon. The contradiction means xy and yz are edges in Δ.

Since the angle at y is even, $m(x) = m(z)$ holds. It implies that the sum of angles at x, y and z is $\geq 2\beta_0 + 4\alpha_0$. Since the fourth angle of Q is $\geq \alpha_0$ and $A_0 = \pi - (\alpha_0 + \beta_0 + \gamma_0) \geq 2\pi/3 - \alpha_0 - \beta_0$, Proposition 5.3 (2) implies $2\pi \geq 5\alpha_0 + 2\beta_0 + m(2\pi/3 - \alpha_0 - \beta_0)$. By using $\alpha_0 \leq \pi/4$ and analyzing all the cases, one concludes that $n(Q) \leq 3$. Here we have a contradiction, as one can check that the union of three chambers can nerve be a quadrilateral with three even angles.

□

6. **Geodesics at different sides.** In this section we discuss a condition ("being at different sides") on two geodesics that in most cases guarantees the two geodesics intersect. This notion is defined in Definition 1.5. In Section 6.1 we give sufficient conditions for two geodesics to be at different sides. In Section 6.2 we show that in most cases two geodesics at different sides must intersect.

6.1. *Criterion for geodesics to be at different sides.* Recall for each vertex v of Δ, the link $\text{Link}(\Delta, v)$ is a rank two spherical building.

Definition 6.1. Let $\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$ be such that the geodesics $\xi_1\xi_2$, $\eta_1\eta_2$ are contained in the 1-skeleton and intersect at a single point v. We say $\xi_1\xi_2$ *and* $\eta_1\eta_2$ *are locally contained in an apartment* if there is an apartment in $\text{Link}(\Delta, v)$ that contains the four directions at v induced by $\xi_1\xi_2$ and $\eta_1\eta_2$.

Proposition 6.2. *Suppose two geodesics $\xi_1\xi_2$ and $\eta_1\eta_2$ are locally contained in an apartment and meet at a vertex v. Then $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides if any one of the following conditions holds:*
(1) *R is not a right triangle;*
(2) *$R \neq (2, 3, 8)$ and $\xi_1\xi_2$ and $\eta_1\eta_2$ make a right angle at v.*

The proof of Proposition 6.2 is contained in the following two lemmas.

Notice that if p_1, p_2 are two vertices in a generalized polygon opposite to the same vertex p, then p_1 and p_2 are of the same type. It follows that if two geodesics $y_1y_2, y_1y_3 \subset \Delta^{(1)}$ intersect in a nontrivial segment y_1x (x lies in the interior of y_1y_2, y_1y_3), then xy_2 and xy_3 make a nonzero even angle at x.

Lemma 6.3. *Suppose two geodesics $\xi_1\xi_2$ and $\eta_1\eta_2$ are locally contained in an apartment. Then $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides if R is not a right triangle.*

Proof. Suppose R is not a right triangle. Let $v = \xi_1\xi_2 \cap \eta_1\eta_2$. Assume $\xi_i\eta_2 \cap \eta_1\eta_2$ is a ray $x\eta_2$ ($x \in \eta_1\eta_2$) for some $i = 1, 2$. Then $\xi_i\eta_2 \subset \Delta^{(1)}$. Now both $\xi_j\xi_i$ ($j \neq i$) and $\eta_2\xi_i$ are contained in $\Delta^{(1)}$, and the distance between them is asymptotically 0. It follows that $\xi_j\xi_i \cap \eta_2\xi_i$ is a ray $y\xi_i$ ($y \in \xi_1\xi_2$), and (v, x, y) is a triangle with an even angle at x, contradicting to Proposition 5.2. Therefore $\xi_i\eta_2 \cap \eta_1\eta_2 = \phi$ for $i = 1, 2$. Similarly for each $y \in \xi_1\xi_2$, $y \neq v$, we have $y\eta_2 \cap \eta_1\eta_2 = \phi$.

Continuity shows that each $y \in [\xi_1, \xi_2] := \xi_1\xi_2 \cup \{\xi_1, \xi_2\}$, $y \neq v$ has a neighborhood U in $[\xi_1, \xi_2] - \{v\}$ such that all the points in U lie at the same side of η_2. The connectivity of $(v, \xi_i] := (v\xi_i - \{v\}) \cup \{\xi_i\}$ implies that all the points in $(v, \xi_i]$ lie at the same side of η_2. On the other hand, $\xi_1\xi_2$ and $\eta_1\eta_2$ are locally contained in an apartment. It is not hard to see that, if y, z lie in different components of $\xi_1\xi_2 - \{v\}$ and are sufficiently close to v, then y and z lie at different sides of η_2. It follows that ξ_1 and ξ_2 lie at different sides of η_2. Similarly ξ_1 and ξ_2 lie at different sides of η_1, and η_1 and η_2 lie at different sides of ξ_i ($i = 1, 2$).

□

Lemma 6.4. *Suppose two geodesics $\xi_1\xi_2$ and $\eta_1\eta_2$ are locally contained in an apartment and meet at a vertex v. Then $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides if $R \neq (2,3,8)$ and $\xi_1\xi_2$ and $\eta_1\eta_2$ make a right angle at v.*

Proof. Suppose $R \neq (2,3,8)$ and $\xi_1\xi_2$ and $\eta_1\eta_2$ make a right angle at v. Assume there is some $q \in \xi_1\xi_2$, $q \neq v$ such that $q\eta_2 \cap \eta_1\eta_2$ is a ray $x\eta_2$. Then $(v,q,x) \subset \Delta^{(1)}$ is a triangle with a right angle at v and an even angle at x, contradicting to Proposition 5.2. One continues to argue as in the proof of Lemma 6.3 that $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides.

□

Remark 6.5. In the case when both R_1 and R_2 have at least 5 edges it is possible to understand the proof of Theorem 1.3 without having to go through too much details, if one is willing to assume Propositions 5.1, 5.2 and 5.3. Here is how to do so. Read Section 2, Section 3, Section 5 (skip Corollaries 5.5, 5.6), Section 6.1 (skip Lemma 6.4), Lemma 6.6 of Section 6.2, Section 7 (skip Lemmas 7.7, 7.8 and Proposition 7.12) and the following:

Let Δ be a Fuchsian building whose chamber R is a k-gon with $k \geq 5$. Then Proposition 5.2 and Corollary 5.4 imply that no triangle or quadrilateral in $\Delta^{(1)}$ can be homeomorphic to a circle.

We claim that if two geodesics are at different sides then they intersect. Suppose $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$) are disjoint and at different sides. Then Lemma 6.6 implies that for each $i = 1,2$ there are vertices $x_i \in \xi_1\xi_2$, $y_i \in \eta_1\eta_2$ such that $x_i\eta_i \cap \eta_1\eta_2 = x'_i\eta_i$ and $y_i\xi_i \cap \xi_1\xi_2 = y'_i\xi_i$ are rays. Since no triangle or quadrilateral in $\Delta^{(1)}$ can be homeomorphic to a circle, we have $x_1 = x_2 = y'_1 = y'_2$ and $x'_1 = x'_2 = y_1 = y_2$. It follows that $x_1x'_1$ makes an angle π with the two rays $\xi_1\xi_2, \eta_1\eta_2$ and hence $\xi_1\eta_1 = x_1\xi_1 \cup x_1x'_1 \cup x'_1\eta_1 \subset \Delta^{(1)}$, contradicting to the definition of being at different sides.

Proof of condition (1) in Lemma 7.3 when R_1 and R_2 have at least 5 edges: Let $c_1, c_2 \subset A^{(1)}$ be two geodesics through v. c_1, c_2 are clearly at different sides. Lemma 7.2 implies c'_1, c'_2 are at different sides. The preceding paragraph shows c'_1 and c'_2 intersect at a vertex $w \in \Delta_2$. Now let $c_3 \subset A^{(1)}$ be any other geodesic through v. Then $w_1 := c'_1 \cap c'_3$ and $w_2 := c'_2 \cap c'_3$ are vertices. If $w \notin c'_3$, then $(w, w_1, w_2) \subset \Delta_2^{(1)}$ is homeomorphic to a circle, contradicting to the above observation.

6.2. *Intersection of geodesics at different sides.* In this section we show, in most cases, that geodesics at different sides must intersect with each other.

The proof of Lemma 6.3 also shows the following:

Lemma 6.6. *If $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$) are disjoint and at different sides, then for each $i = 1,2$ there are vertices $x_i \in \xi_1\xi_2$, $y_i \in \eta_1\eta_2$ such that $x_i\eta_i \cap \eta_1\eta_2$ and $y_i\xi_i \cap \xi_1\xi_2$ are rays.*

□

Proposition 6.7. *Suppose $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$) are disjoint and at different sides. Let $y_i \in \eta_1\eta_2$ ($i = 1,2$) be a vertex such that $y_i\xi_i \cap \xi_1\xi_2 = y'_i\xi_i$ is a ray and $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. If $R \neq (2,3,8)$, then one of the following holds:*
(1) $y'_1 \in$ interior$(y'_2\xi_1)$ and $y_1 \neq y_2$;
(2) $y'_1 = y'_2$ and $y_1 \neq y_2$.

Proof. Since Δ is a CAT(-1) space, there is a unique geodesic segment between two points. We first notice $y'_1 \in$ interior$(y'_2\xi_2)$ cannot happen, since otherwise $y_1y'_1 \cup y'_1y'_2 \cup y'_2y_2$ would be a geodesic segment connecting $y_1 \in \eta_1\eta_2$ and $y_2 \in \eta_1\eta_2$, but different from $y_1y_2 \subset \eta_1\eta_2$. Therefore $y'_1 \in y'_2\xi_1$.

Suppose $y'_1 \neq y'_2$, that is, $y'_1 \in$ interior$(y'_2\xi_1)$. Notice that the two angles $\angle_{y'_1}(y_1, \xi_2)$ and $\angle_{y'_2}(y_2, \xi_1)$ are even. If $y_1 = y_2$, then (y_1, y'_1, y'_2) is a triangle with two even angles, contradicting to Proposition 5.2 and the assumption that $R \neq (2,3,8)$. Therefore $y_1 \neq y_2$ and (1) holds.

Now suppose $y_1' = y_2'$. We need to show $y_1 \neq y_2$. Suppose $y_1 = y_2$. Let $x_i \in \xi_1\xi_2$ $(i = 1, 2)$ be a vertex such that $x_i\eta_i \cap \eta_1\eta_2 = x_i'\eta_i$ is a geodesic ray and $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$. Assume $x_i' = y_1$ for some $i = 1, 2$. Then the uniqueness of geodesic implies $x_i = y_1'$. It follows that $\eta_i\xi_1 = \eta_i y_1 \cup y_1 y_1' \cup y_1'\xi_1 \subset \Delta^{(1)}$ and $\eta_i\xi_1 \cap \xi_2\xi_1$ is a ray, contradicting to the assumption that $\xi_1\xi_2$ and $\eta_1\eta_2$ are at different sides. Therefore $x_i' \neq y_1$ for $i = 1, 2$. Notice $x_i' \in \text{interior}(y_1\eta_i)$, otherwise $x_i x_i' \cup x_i' y_1$ and $x_i y_1' \cup y_1' y_1$ would be two distinct geodesic segments from x_i to y_1. At least one of the two angles $\angle_{y_1}(x_1', y_1')$, $\angle_{y_1}(x_2', y_1')$ is $\geq \pi/2$. We may assume $\angle_{y_1}(x_1', y_1') \geq \pi/2$. Then $x_1 y_1 = x_1 y_1' \cup y_1' y_1$ and (x_1, y_1, x_1') is a triangle with an even angle at x_1' and another angle $\angle_{y_1}(x_1', x_1) = \angle_{y_1}(x_1', y_1') \geq \pi/2$. Proposition 5.2 implies $R = (2, 3, 8)$, contradicting to our assumption.

□

Proposition 6.8. *Suppose $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ $(\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta)$ are disjoint and at different sides. Let $x_1 \in \xi_1\xi_2$, $y_1 \in \eta_1\eta_2$ be vertices such that $x_1\eta_1 \cap \eta_1\eta_2 = x_1'\eta_1$ and $y_1\xi_1 \cap \xi_1\xi_2 = y_1'\xi_1$ are rays and $x_1\eta_1 \cap \xi_1\xi_2 = \{x_1\}$, $y_1\xi_1 \cap \eta_1\eta_2 = \{y_1\}$. Then after possibly switching ξ with η and x with y, one of the following holds:*
(1) *$y_1' \in$ interior$(x_1\xi_1)$ and $y_1 \in$ interior$(x_1'\eta_1)$;*
(2) *$y_1' \in$ interior$(x_1\xi_1)$ and $y_1 = x_1'$;*
(3) *$y_1' \in$ interior$(x_1\xi_1)$ and $x_1' \in$ interior$(y_1\eta_1)$.*

Proof. Assume $x_1 = y_1'$ and $x_1' = y_1$. Then $\xi_1\eta_1 = \xi_1 y_1' \cup y_1' y_1 \cup y_1\eta_1$, which implies $\xi_1\eta_1 \cap \xi_1\xi_2 = y_1'\xi_1$ is a ray, contradicting to the assumption that η_1 and η_2 lie at different sides of ξ_1. Hence either $x_1 \neq y_1'$ or $x_1' \neq y_1$. We assume $x_1 \neq y_1'$. The case $x_1' \neq y_1$ can be handled similarly.

Suppose $x_1 \neq y_1'$ and $x_1' = y_1$. Then either $y_1' \in \text{interior}(x_1\xi_1)$ or $x_1 \in \text{interior}(y_1'\xi_1)$. Since $y_1\xi_1 \cap \xi_1\xi_2 = y_1'\xi_1$, the uniqueness of geodesic implies that $x_1 \in \text{interior}(y_1'\xi_1)$ cannot happen and we have (2).

Suppose $x_1 \neq y_1'$ and $x_1' \neq y_1$. There are 4 possibilities:
(a) $y_1' \in \text{interior}(x_1\xi_1)$ and $y_1 \in \text{interior}(x_1'\eta_1)$;
(b) $y_1' \in \text{interior}(x_1\xi_1)$ and $x_1' \in \text{interior}(y_1\eta_1)$;
(c) $x_1 \in \text{interior}(y_1'\xi_1)$ and $x_1' \in \text{interior}(y_1\eta_1)$;
(d) $x_1 \in \text{interior}(y_1'\xi_1)$ and $y_1 \in \text{interior}(x_1'\eta_1)$.
Case (d) cannot happen since otherwise there are two different geodesic segments $y_1 y_1' \cup y_1' x_1$ and $y_1 x_1' \cup x_1' x_1$ from y_1 to x_1, a contradiction. (a) corresponds to (1) and (b) corresponds to (3). (c) corresponds to (1) after switching x with y and ξ with η.

□

Proposition 6.9. *Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ $(\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta)$ be two disjoint geodesics. If $\xi_1\xi_2, \eta_1\eta_2$ are at different sides, then R is a right triangle.*

We need some lemmas.

Lemma 6.10. *Suppose $\xi_1\xi_2, \eta_1\eta_2$ are disjoint and at different sides. Let $x_1 \in \xi_1\xi_2$, $y_1 \in \eta_1\eta_2$ be vertices such that $x_1\eta_1 \cap \eta_1\eta_2 = x_1'\eta_1$ and $y_1\xi_1 \cap \xi_1\xi_2 = y_1'\xi_1$ are rays and $x_1\eta_1 \cap \xi_1\xi_2 = \{x_1\}$, $y_1\xi_1 \cap \eta_1\eta_2 = \{y_1\}$. If R is not a right triangle, then Proposition 6.8* (3) *holds.*

Proof. We prove by contradiction that Proposition 6.8 (1) and (2) cannot happen. Since R is not a right triangle, Proposition 5.2 implies there is no triangle contained in the 1-skeleton that has a nonzero even angle. If Proposition 6.8 (2) holds, then (x_1, y_1', y_1) is a triangle with a nonzero even angle $\angle_{y_1'}(x_1, y_1)$, a contradiction. Suppose Proposition 6.8 (1) holds. Then $x_1 y_1 = x_1 x_1' \cup x_1' y_1$ and again (x_1, y_1', y_1) is a triangle with a nonzero even angle $\angle_{y_1'}(x_1, y_1)$, a contradiction. Finally we notice that Proposition 6.8 (3) remains the same after we switch x with y and ξ with η.

□

Lemma 6.11. *Suppose $\xi_1\xi_2, \eta_1\eta_2$ are disjoint and at different sides. Let $y_i \in \eta_1\eta_2$ $(i = 1, 2)$ be a vertex such that $y_i\xi_i \cap \xi_1\xi_2 = y_i'\xi_i$ is a ray and $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. If R is not a right triangle, then Proposition 6.7 (1) holds.*

Proof. Assume Proposition 6.7 (2) holds, that is, $y_1' = y_2'$ and $y_1 \neq y_2$. By Lemma 6.10, $y_1' \in \text{interior}(x_1\xi_1)$ and $x_1' \in \text{interior}(y_1\eta_1)$. If $y_2 \in \text{interior}(x_1'\eta_2)$, then $x_1y_2 = x_1y_1' \cup y_1'y_2$ and (x_1, y_2, x_1') is a triangle with a nonzero even angle $\angle_{x_1'}(x_1, y_2)$, contradicting to Proposition 5.2. If $y_2 \in x_1'\eta_1$, then $x_1y_1' \cup y_1'y_2$ and $x_1x_1' \cup x_1'y_2$ are two distinct geodesic segments from x_1 to y_2, a contradiction. □

Proof of Proposition 6.9. Suppose R is not a right triangle. By Lemma 6.6 there are vertices $y_i \in \eta_1\eta_2$ $(i = 1, 2)$ such that $y_1\xi_1 \cap \xi_1\xi_2 = y_1'\xi_1$ and $y_2\xi_2 \cap \xi_1\xi_2 = y_2'\xi_2$ are rays. We may assume $y_1\xi_1 \cap \eta_1\eta_2 = \{y_1\}$, $y_2\xi_2 \cap \eta_1\eta_2 = \{y_2\}$. Lemma 6.11 implies $y_1' \in \text{interior}(y_2'\xi_1)$ and $y_1 \neq y_2$. Let $\bar\eta \neq \bar{\bar\eta} \in \{\eta_1, \eta_2\}$ such that $y_1 \in \mathit{interior}(y_2\bar\eta)$. Let $\bar x \in \xi_1\xi_2$ be a vertex such that $\bar x\bar\eta \cap \bar\eta\bar{\bar\eta} = \bar x'\bar\eta$ is a ray and $\bar x\bar\eta \cap \xi_1\xi_2 = \{\bar x\}$. Lemma 6.10 implies the six points $\{y_1, y_1', y_2, y_2', \bar x, \bar x'\}$ are pairwise distinct. Applying Lemma 6.10 to the rays $y_1\xi_1$ and $\bar x\bar\eta$ we see $\bar x \in \text{interior}(y_1'\xi_2)$. Applying Lemma 6.10 again to the rays $y_2\xi_2$ and $\bar x\bar\eta$ we see $\bar x \in \text{interior}(y_2'\xi_1)$. It follows that $\bar x \in \text{interior}(y_1'y_2')$. Similarly, if $\bar{\bar x} \in \xi_1\xi_2$ is a vertex such that $\bar{\bar x}\bar{\bar\eta} \cap \bar\eta\bar{\bar\eta} = \bar{\bar x}'\bar{\bar\eta}$ is a ray, then $\bar{\bar x} \in \text{interior}(y_1'y_2')$. Lemma 6.11 implies $\bar{\bar x} \neq \bar x$. Therefore $y_1'y_2'$ contains at least two vertices in its interior. Notice $(y_1', y_1, y_2, y_2') \subset \Delta^{(1)}$ is a quadrilateral with even angles at two adjacent corners y_1' and y_2'. Now Corollary 5.5 implies that $R = (2, 3, 8)$, contradicting to our assumption. □

The following lemma follows easily from the definition.

Lemma 6.12. *Let $\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$ with $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$. Suppose $\xi_1\xi_2, \eta_1\eta_2$ are at different sides and $\xi_1\xi_2 \cap \eta_1\eta_2 \neq \phi$, then $\xi_1\xi_2 \cap \eta_1\eta_2$ is a vertex.*

7. **When R_1, R_2 are not right triangles.** In this section we prove Theorem 1.3 in the case when the chambers are not right triangles.

7.1. *Criterion for extension.* In this section we find conditions that ensure a homeomorphism $h : \partial\Delta_1 \to \partial\Delta_2$ extends to an isomorphism $\Delta_1 \to \Delta_2$.

Let Δ_1 and Δ_2 be Fuchsian buildings with chambers R_1 and R_2 respectively, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. For any $\xi \in \partial\Delta_1$, let $\xi' = h(\xi) \in \partial\Delta_2$. For $\xi, \eta \in \partial\Delta_1$, we call $\xi'\eta'$ the *image* of $\xi\eta$.

Since nonempty open subsets in $\partial\Delta_1$ and $\partial\Delta_2$ have positive measures and h preserves the combinatorial cross ratio a.e., Lemma 3.11 implies the following result.

Lemma 7.1. *For any $\xi, \eta \in \partial\Delta_1$, $\xi\eta$ lies in the 1-skeleton of Δ_1 if and only if its image lies in the 1-skeleton of Δ_2. Furthermore, any edge in $\xi\eta$ is contained in exactly $q + 1$ chambers if and only if any edge in $\xi'\eta'$ is contained in exactly $q + 1$ chambers.*

Lemma 7.1 generalizes Lemma 2.4.8 of [Bo1]. It implies that $h(B_1) = B_2$, where B_i $(i = 1, 2)$ is the set of regular points in $\partial\Delta_i$. The following crucial lemma follows easily from Lemma 7.1 and Lemma 3.12.

Lemma 7.2. *Let $\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta_1$ with $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta_1^{(1)}$. Then $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta_1$ are at different sides if and only if $\xi_1'\xi_2', \eta_1'\eta_2' \subset \Delta_2$ are at different sides.*

For any vertex $v \in \Delta_1$, let $\mathcal{D}_v$ be the family of geodesics passing through v and contained in $\Delta_1^{(1)}$, and $\mathcal{D}_v'$ the family of images of the geodesics in $\mathcal{D}_v$. For each apartment A containing v, let $\mathcal{D}_{A,v}$ be the set of geodesics passing through v and contained in $A^{(1)}$, and $\mathcal{D}_{A,v}'$ the set of images of the geodesics in $\mathcal{D}_{A,v}$. We shall prove that for any vertex $v \in \Delta_1$, the geodesics in $\mathcal{D}_v'$ intersect in a unique vertex of Δ_2.

Lemma 7.3. *Let $v \in \Delta_1$ be a vertex. Suppose the following conditions hold:*
(1) for any apartment A containing v, the geodesics in $\mathcal{D}'_{A,v}$ intersect in a unique vertex $v_A \in \Delta_2$;
(2) $v_{A_1} = v_{A_2}$ holds for any two apartments $A_1, A_2 \subset \Delta_1$ containing v with $Link(A_1, v) \cap Link(A_2, v)$ a half apartment in $Link(\Delta_1, v)$.
Then the geodesics in $\mathcal{D}'_v$ intersect in a unique vertex.

Proof. Let $c, c' \subset \Delta_1^{(1)}$ be two geodesics that pass through v. Let A and A' be two apartments that contain c and c' respectively. Lemma 2.2 implies there is a sequence of apartments containing v: $A_0 = A, \cdots, A_n = A'$ such that $\text{Link}(A_i, v) \cap \text{Link}(A_{i+1}, v)$ $(0 \le i \le n)$ is a half apartment in $\text{Link}(\Delta_1, v)$. Now the lemma follows from the assumptions.

□

Lemma 7.4. *Suppose for any vertex $v \in \Delta_1$, the geodesics in $\mathcal{D}'_v$ intersect in a unique vertex of Δ_2 denoted by $f(v)$, and for any vertex $w \in \Delta_2$, the family of geodesics $\{h^{-1}(\xi)h^{-1}(\eta) : \xi, \eta \in \partial\Delta_2,\ w \in \xi\eta \subset \Delta_2^{(1)}\}$ also intersect in a unique vertex of Δ_1. Then the map $f : \Delta_1^{(0)} \to \Delta_2^{(0)}$ is a bijection that extends to an isomorphism from Δ_1 to Δ_2.*

Proof. The map f is clearly bijective. We claim for any two vertices $v_1, v_2 \in \Delta_1$, v_1 and v_2 are adjacent if and only if $f(v_1)$ and $f(v_2)$ are adjacent. The claim implies f extends to an isomorphism between the 1-skeletons of Δ_1 and Δ_2. It is an exercise to show that f maps the boundary of a chamber in Δ_1 to the boundary of a chamber in Δ_2. Therefore f extends to an isomorphism from Δ_1 to Δ_2. Next we prove the claim.

Let $\xi, \eta \in \partial\Delta_1$ with $\xi\eta \subset \Delta_1^{(1)}$. By the definition of f, $f(v) \in \xi'\eta'$ for any vertex $v \in \xi\eta$. It suffices to show that if $x, y \in \xi\eta$ are two vertices such that $x \in \text{interior}(y\xi)$, then $f(x) \in \text{interior}(f(y)\xi')$. Since the link $\text{Link}(\Delta_1, x)$ is a thick spherical building, there is a an edge $e = xz$ such that $zx \cup x\xi$ is a geodesic ray and $xz \cap xy = \{x\}$. We extend the ray $zx \cup x\xi$ to obtain a complete geodesic $\xi\xi_1 \subset \Delta_1^{(1)}$. We note $x \in \xi\xi_1$ and $y \notin \xi\xi_1$. The images $\xi'\eta'$ and $\xi'\xi'_1$ of $\xi\eta$ and $\xi\xi_1$ are contained in the 1-skeleton of Δ_2 and have a common point at infinity. Since Δ_2 is CAT(-1), the intersection $\xi'\eta' \cap \xi'\xi'_1$ is a ray asymptotic to ξ'. The assumption and the fact that $x \in \xi\eta \cap \xi\xi_1$ imply $f(x) \in \xi'\eta' \cap \xi'\xi'_1$. If $f(y) \in \text{interior}(f(x)\xi')$, then $f(y) \in \xi'\eta' \cap \xi'\xi'_1$, which in turn implies $y \in \xi\eta \cap \xi\xi_1$, contradicting to $y \notin \xi\xi_1$.

□

By Lemma 7.3 and Lemma 7.4, to prove Theorem 1.3 we only need to verify the two conditions in Lemma 7.3.

7.2. *Proof of Theorem 1.3 when R_1, R_2 are not right triangles.* In this section we shall show that the conditions in Lemma 7.3 are satisfied when R_1, R_2 are not right triangles.

Lemma 7.5. *Let Δ be a Fuchsian building, and $\xi_1, \xi_2, \xi_3 \in \partial\Delta$ be three points such that $\xi_1\xi_2$, $\xi_2\xi_3$ and $\xi_3\xi_1$ all lie in $\Delta^{(1)}$. Then exactly one of the following occurs:*
(1) there exists a vertex $v \in \Delta$ such that $\xi_1\xi_2 = v\xi_1 \cup v\xi_2$, $\xi_1\xi_3 = v\xi_1 \cup v\xi_3$ and $\xi_2\xi_3 = v\xi_2 \cup v\xi_3$;
(2) $R = (2, 3, 8)$ and there are vertices v_1, v_2, v_3 such that $\xi_1\xi_2 \cap \xi_1\xi_3 = v_1\xi_1$, $\xi_2\xi_1 \cap \xi_2\xi_3 = v_2\xi_2$, $\xi_1\xi_3 \cap \xi_2\xi_3 = v_3\xi_3$; furthermore, $T := (v_1, v_2, v_3)$ has three even angles and $S(T)$ is as shown in Figure 3 (h).

Proof. Since Δ is CAT(-1) and $\xi_1\xi_2$, $\xi_2\xi_3$ $\xi_3\xi_1$ are all contained in the 1-skeleton, the intersections $\xi_1\xi_2 \cap \xi_1\xi_3$, $\xi_1\xi_2 \cap \xi_2\xi_3$ and $\xi_1\xi_3 \cap \xi_2\xi_3$ are all rays. Let v be the vertex with $\xi_1\xi_2 \cap \xi_1\xi_3 = v\xi_1$. If $v \in \xi_2\xi_3$, then the convexity of geodesic implies $\xi_2\xi_3 = v\xi_2 \cup v\xi_3$ and so (1) holds. Suppose $v \notin \xi_2\xi_3$. Then there are vertices $v_2 \in \text{interior}(v\xi_2)$, $v_3 \in \text{interior}(v\xi_3)$ such that $\xi_1\xi_2 \cap \xi_2\xi_3 = v_2\xi_2$ and $\xi_1\xi_3 \cap \xi_2\xi_3 = v_3\xi_3$. Now $T = (v, v_2, v_3) \subset \Delta^{(1)}$ is a triangle with three even angles. Proposition 5.2 implies that $R = (2, 3, 8)$ and $S(T)$ is as shown in Figure 3 (h), hence (2) holds.

□

Remark 7.6. In the above proof we came up with a triangle $T \subset \Delta^{(1)}$ which has three even angles. We then check the triangles listed in Proposition 5.2 and find out that only one of them has this property and the chamber has to be $(2,3,8)$. We shall use Proposition 5.2 very often in this paper. Usually a triangle $T \subset \Delta^{(1)}$ is present which has a certain property, for example, "has an even angle", "one side contains one or two vertices in the interior", "has an angle $> \pi/2$". When we apply Proposition 5.2, we check the triangles listed in Proposition 5.2 to see whether any of them or which ones of them have the stated property. This is how we apply Proposition 5.2.

Recall for each vertex v in a Fuchsian building Δ, the link $\mathrm{Link}(\Delta, v)$ is a generalized m-gon for some $m \in \{2,3,4,6,8\}$. We define $m(v) = m$ if $\mathrm{Link}(\Delta, v)$ is a generalized m-gon.

Lemma 7.7. *Suppose the following condition holds:*
for any two geodesics $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta_1$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta_1$) that are locally contained in an apartment and are at different sides, their images $\xi_1'\xi_2'$, $\eta_1'\eta_2'$ have nonempty intersection.
If R_2 is not a right triangle, then for any vertex $v \in \Delta_1$ with $m(v) \neq 3$ and any apartment $A \ni v$, the geodesics in $\mathcal{D}'_{A,v}$ intersect in a unique vertex v_A.

Proof. We observe that for any two geodesics $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta_1$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta_1$) that are locally contained in an apartment and are at different sides, the intersection $\xi_1'\xi_2' \cap \eta_1'\eta_2'$ is a single point. Otherwise $\xi_1'\xi_2' \cap \xi_1'\eta_i'$ is a ray for some $i = 1,2$, which implies that $\xi_1\xi_2 \cap \xi_1\eta_i$ is a ray, contradicting to the fact that $\xi_1\xi_2$ and $\eta_1\eta_2$ are at different sides.

The case $m(v) = 2$ follows from the assumption. Assume $m(v) \geq 4$. First suppose there are three members c_1, c_2, c_3 of $\mathcal{D}'_{A,v}$ intersecting at one point w. Suppose c is a member of $\mathcal{D}'_{A,v}$ that does not contain w. Then the three intersection points $p_i = c_i \cap c (i = 1,2,3)$ are pairwise distinct. We may assume p_2 lies between p_1 and p_3. Then $(w, p_1, p_3) \subset \Delta_2^{(1)}$ is a triangle that is not the boundary of a chamber, contradicting to Proposition 5.2.

Now suppose the intersection of any three members of $\mathcal{D}'_{A,v}$ is empty. Fix some $c \in \mathcal{D}'_{A,v}$. Since $m(v) \geq 4$, there are at least three members c_1, c_2, c_3 of $\mathcal{D}'_{A,v}$ that are distinct from c. Let $p_i = c \cap c_i$ ($i = 1,2,3$). We may assume p_2 lies between p_1 and p_3. Let $w = c_1 \cap c_3$. We obtain a contradiction as above.

□

Lemma 7.8. *Suppose the following condition holds:*
for any two geodesics $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta_1$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta_1$) that are locally contained in an apartment and are at different sides, their images $\xi_1'\xi_2'$, $\eta_1'\eta_2'$ have nonempty intersection.
If R_2 is not a right triangle, then for any vertex $v \in \Delta_1$ with $m(v) = 3$ and any apartment $A \ni v$, the geodesics in $\mathcal{D}'_{A,v}$ intersect in a unique vertex v_A.

Proof. Let $\xi_i\eta_i \subset A^{(1)}$ ($\xi_i, \eta_i \in \partial A$, $i = 1,2,3$) be the three geodesics that pass through v. We may assume $\angle_v(\xi_i, \xi_j) = 2\pi/3$ for $i \neq j$. Note $\mathrm{Link}(\Delta_1, v)$ is a thick generalized 3-gon, $\mathrm{Link}(A, v)$ is an apartment in $\mathrm{Link}(\Delta_1, v)$ and the initial directions of $v\xi_1$, $v\xi_2$, $v\xi_3$ are of the same type. Proposition 2.3 implies that there is an edge $vv' \subset \Delta_1$ such that $v'v \cup v\xi_i$ ($i = 1,2,3$) is still a geodesic. We extend vv' to a ray $v\xi \supset vv'$ ($\xi \in \partial\Delta_1$). Then $\xi\xi_i = v\xi \cup v\xi_i$ for $i = 1,2,3$.

By assumption, $\xi_1'\eta_1'$ and $\xi_2'\eta_2'$ intersect at some vertex w. Assume exactly one of $\xi_1'\xi'$, $\xi_2'\xi'$ contains w, say $w \in \xi_1'\xi'$ and $w \notin \xi_2'\xi'$. Then $\xi'\xi_1' = w\xi' \cup w\xi_1'$, and $\xi_2'\xi' \cap \xi_2'\eta_2' = x\xi_2'$ and $\xi'\xi_2' \cap \xi'\xi_1' = y\xi'$ for some $x \in \mathrm{interior}(w\xi_2')$, $y \in \mathrm{interior}(w\xi')$. It follows that $(w, x, y) \subset \Delta_2^{(1)}$ is a triangle with two even angles $\angle_x(w, y)$, $\angle_y(w, x)$. Proposition 5.2 implies $R_2 = (2,3,8)$, contradicting to our assumption. Therefore either $w \in \xi_1'\xi' \cap \xi_2'\xi'$ or $w \notin \xi_1'\xi'$, $w \notin \xi_2'\xi'$.

Assume none of $\xi_1'\xi'$, $\xi_2'\xi'$ contains w. Then there are $x \in \mathrm{interior}(w\xi_2')$, $y \in \mathrm{interior}(w\xi_1')$ such that $\xi'\xi_2' \cap \xi_2'\eta_2' = x\xi_2'$, $\xi'\xi_1' \cap \xi_1'\eta_1' = y\xi_1'$. Note $y\xi' \cap \xi_2'\eta_2' = \phi$, otherwise if $p = y\xi' \cap \xi_2'\eta_2'$ then $(w, y, p) \subset \Delta_2^{(1)}$ is a triangle with an even angle $\angle_y(w, p)$, contradicting to Proposition 5.2. In particular, $x \notin y\xi' \subset \xi_1'\xi'$. It follows that $\xi'\xi_1' \cap \xi'\xi_2' = z\xi'$ for some $z \in \mathrm{interior}(x\xi')$. Similar

argument shows $z \in \text{interior}(y\xi')$. Now $(z, y, w, x) \subset \Delta_2^{(1)}$ is a quadrilateral with three even angles $\angle_x(w, z)$, $\angle_y(w, z)$ and $\angle_z(x, y)$, contradicting to Corollary 5.6. Therefore both $\xi_1'\xi'$ and $\xi_2'\xi'$ contain w.

Assume $w \notin \xi_3'\eta_3'$. Then $\xi_1'\eta_1'$ and $\xi_3'\eta_3'$ intersect at some vertex $w' \neq w$. Then the above argument shows that both $\xi_1'\xi'$ and $\xi_3'\xi'$ contain w'. In particular, $\xi_1'\xi'$ contains both w and w'. We have either $w' \in \text{interior}(w\xi')$ or $w \in \text{interior}(w'\xi')$. First suppose $w' \in \text{interior}(w\xi')$. Then $w' \in \text{interior}(w\eta_1')$. It follows that $\angle_w(\xi_2', \eta_1') = \angle_w(\xi_2', w') = \angle_w(\xi_2', \xi')$, which is π since $\xi_2'\xi'$ contains w. Hence $\xi_2'\eta_1' = w\xi_2' \cup w\eta_1' \subset \Delta_2^{(1)}$. Then $\xi_2\eta_1$ must be a geodesic contained in the 1-skeleton of Δ_1, which is not true since $\xi_1\eta_1$ and $\xi_2\eta_2$ are contained in the apartment A. Similarly one obtains a contradiction if $w \in \text{interior}(w'\xi')$.

□

Lemma 7.2 and Proposition 6.9 imply that the condition in Lemma 7.7 and Lemma 7.8 is satisfied when R_2 is not a right triangle. It follows that condition (1) of Lemma 7.3 holds when R_2 is not a right triangle. When R_1, R_2 are not right triangles, Theorem 1.3 follows from Proposition 7.9, Lemma 7.3 and Lemma 7.4.

Proposition 7.9. *Suppose condition* (1) *of Lemma 7.3 is satisfied and R_1, R_2 are not right triangles. Then condition* (2) *of Lemma 7.3 is also satisfied.*

We prove Proposition 7.9 in two lemmas depending on whether $m(v) = 3$.

Lemma 7.10. *Suppose condition* (1) *of Lemma 7.3 is satisfied and R_1, R_2 are not right triangles. Then condition* (2) *of Lemma 7.3 holds for those v with $m(v) \neq 3$.*

Proof. Let $v \in \Delta_1$ be a vertex with $m(v) \neq 3$ and $A_1, A_2 \subset \Delta_1$ two apartments containing v with $\text{Link}(A_1, v) \cap \text{Link}(A_2, v)$ a half apartment in $\text{Link}(\Delta_1, v)$. Let $a, b \in \text{Link}(\Delta_1, v)$ be the two opposite vertices of the half apartment $\text{Link}(A_1, v) \cap \text{Link}(A_2, v)$, and c the midpoint of $\text{Link}(A_1, v) \cap \text{Link}(A_2, v)$. Note c is a vertex as $m(v) \neq 3$. Denote by c_i $(i = 1, 2)$ the vertex in $\text{Link}(A_i, v)$ opposite to c. The points $\xi_i, \eta_i, \omega_i, \zeta_i \in \partial A_i (i = 1, 2)$ are defined as follows: a is the initial direction of $v\xi_i$, b is the initial direction of $v\eta_i$, c is the initial direction of $v\omega_i$ and c_i is the initial direction of $v\zeta_i$. Note $\omega_i\zeta_j = v\omega_i \cup v\zeta_j$ and $\zeta_1\zeta_2 = v\zeta_1 \cup v\zeta_2$.

Note $\zeta_1\zeta_2$ and $\xi_1\eta_1$ are locally contained in an apartment. Since R_1 is not a right triangle, Lemma 6.3 implies $\zeta_1\zeta_2$ and $\xi_1\eta_1$ are at different sides. It follows that $\zeta_1'\zeta_2'$ and $\xi_1'\eta_1'$ are at different sides. Since R_2 is not a right triangle, Proposition 6.9 implies $\zeta_1'\zeta_2' \cap \xi_1'\eta_1' \neq \phi$. Similarly the following holds: $\zeta_1'\zeta_2' \cap \xi_2'\eta_2' \neq \phi$, $\zeta_2'\omega_i' \cap \xi_1'\eta_1' \neq \phi$ and $\xi_2'\eta_2' \cap \omega_1'\zeta_i' \neq \phi$ $(i = 1, 2)$.

Let $w = \xi_1'\eta_1' \cap \zeta_1'\omega_1'$. We notice $w = v_{A_1}$. Since $R_2 \neq (2, 3, 8)$, Lemma 7.5 applied to the three points $\omega_1', \zeta_1', \zeta_2'$ shows that there is some $w' \in \omega_1'\zeta_1'$ such that $\zeta_1'\zeta_2' = w'\zeta_1' \cup w'\zeta_2'$ and $\omega_1'\zeta_2' = w'\omega_1' \cup w'\zeta_2'$. Assume $w' \in \text{interior}(w\zeta_1')$. Then w is the only intersection point of $\zeta_i'\omega_1' (i = 1, 2)$ with $\xi_1'\eta_1'$. It follows that $\zeta_1'\zeta_2'$ and $\xi_1'\eta_1'$ are disjoint, contradicting to the conclusion in the second paragraph. Similarly $w' \in \text{interior}(w\omega_1')$ is impossible and we conclude $w' = w$. In particular, $w \in \zeta_2'\omega_1'$.

We need to prove $w \in \zeta_2'\omega_2'$. If $\omega_2 = \omega_1$, then we are done. Suppose $\omega_2 \neq \omega_1$, then $\zeta_2\omega_2 \cap \zeta_2\omega_1$ is a ray. It follows that $\zeta_2'\omega_2' \cap \zeta_2'\omega_1' = w_2\zeta_2'$ (for some $w_2 \in \zeta_2'\omega_1'$) is also a ray. Assume $w_2 \in \text{interior}(w\zeta_2')$. By the second paragraph $\zeta_2'\omega_2'$ and $\xi_1'\eta_1'$ intersect at some point $w_1 \in \xi_1'\eta_1'$. Now we have a triangle (w, w_1, w_2) with an even angle $\angle_{w_2}(w, w_1)$, contradicting to Proposition 5.2. Therefore $w \in w_2\zeta_2' \subset \zeta_2'\omega_2'$.

Next we prove $w \in \xi_2'\eta_2'$. By the second paragraph, $\xi_2'\eta_2'$ and $\omega_1'\zeta_1'$ intersect at some point x. If $x \in \text{interior}(w\omega_1')$, then $\xi_2'\eta_2' \cap \zeta_1'\zeta_2' = \phi$, contradicting to the second paragraph. Similarly $x \notin \text{interior}(w\zeta_1')$ and we conclude $w = x \in \xi_2'\eta_2'$.

□

Lemma 7.11. *Suppose condition* (1) *of Lemma 7.3 is satisfied and* R_1, R_2 *are not right triangles. Then condition* (2) *of Lemma 7.3 holds for those* v *with* $m(v) = 3$.

Proof. Let $v \in \Delta_1$ be a vertex with $m(v) = 3$, $A_1, A_2 \subset \Delta_1$ be two apartments containing v with $\mathrm{Link}(A_1, v) \cap \mathrm{Link}(A_2, v)$ a half apartment in $\mathrm{Link}(\Delta_1, v)$. Let $a, b \in \mathrm{Link}(\Delta_1, v)$ be the two opposite vertices of the half apartment $\mathrm{Link}(A_1, v) \cap \mathrm{Link}(A_2, v)$, and c, d the other two vertices on $\mathrm{Link}(A_1, v) \cap \mathrm{Link}(A_2, v)$ such that a and c are adjacent. Let $c_i, d_i \in \mathrm{Link}(A_i, v)$ $(i = 1, 2)$ be vertices such that c_i and c are opposite, and d_i and d are opposite. The points $\xi_i, \eta_i, \omega_i, \tau_i, \zeta_i, \phi_i \in \partial A_i$ $(i = 1, 2)$ are defined as follows: a is the initial direction of $v\xi_i$, b is the initial direction of $v\eta_i$, c is the initial direction of $v\omega_i$, d is the initial direction of $v\tau_i$, c_i is the initial direction of $v\zeta_i$ and d_i is the initial direction of $v\phi_i$.

Note d_2 is opposite to both d and c_1. Hence $\tau_1\phi_2 = v\tau_1 \cup v\phi_2$ and $\zeta_1\phi_2 = v\zeta_1 \cup v\phi_2$. Let $w = \omega_1'\zeta_1' \cap \tau_1'\phi_1'$. Then the proof of Lemma 7.8 shows that $\tau_1'\phi_2' = w\tau_1' \cup w\phi_2'$ and $\zeta_1'\phi_2' = w\zeta_1' \cup w\phi_2'$. It follows that $\zeta_1'\phi_2' \cap \tau_1'\phi_1' = w$. On the other hand, d is opposite to both d_1 and d_2 and we have $\tau_2\phi_i = v\tau_2 \cup v\phi_i$. Apply the arguments in Lemma 7.8 again we see $\tau_2'\phi_i' = w\tau_2' \cup w\phi_i'$. In particular, $w \in \tau_2'\phi_2'$. Similarly $w \in \zeta_2'\omega_2'$. It follows that $v_{A_1} = w = \zeta_2'\omega_2' \cap \tau_2'\phi_2' = v_{A_2}$.

□

Proposition 7.12. *Let* Δ_1 *and* Δ_2 *be two Fuchsian buildings, and* $h : \partial\Delta_1 \to \partial\Delta_2$ *a homeomorphism that preserves the combinatorial cross rario almost everywhere. If one of* R_1, R_2 *is a right triangle, then so is the other.*

Proof. We assume R_1 is a right triangle and R_2 is not, and shall derive a contradiction from this. Let A be an apartment in Δ_1 and C any chamber in A. Denote the vertices of C by x, y, z such that the angle at y is $\pi/2$. Denote the geodesics in A that contain xz, xy and yz respectively by c_1, c_2 and c. Let C' be the chamber in A that shares the edge xy with C. Denote the third vertex of C' by z' and the geodesic in A containing xz' by c_3.

It is clear that any two of the 4 geodesics c, c_1, c_2, c_3 are at different sides. Let c' and c_i' $(i = 1, 2, 3)$ be the images of c and c_i respectively. Lemma 7.2 implies that any two of the 4 geodesics c', c_1', c_2', c_3' are also at different sides. Since R_2 is not a right triangle, Proposition 6.9 implies any two of the 4 geodesics c', c_1', c_2', c_3' have nonempty intersection. Then the argument in the proof of Lemma 7.7 shows that these 4 geodesics have a common point. In particular, the three geodesics c_1', c_2' and c', which are the images of the three geodesics containing the three edges of the chamber C, have a unique common point w_C. We have also shown the following: if two chambers $C_1, C_2 \subset A$ share an edge e and the angle of C_1 at one of the two vertices of e is $\pi/2$, then $w_{C_1} = w_{C_2}$.

Suppose two chambers $C_1, C_2 \subset A$ share an edge $e = v_1v_2$ and both angles of C_1 at the two vertices of e are $\neq \pi/2$. Then at least one of these two angles is $\neq \pi/3$, say the angle at v_1 is $\neq \pi/3$. There are $m(v_1) \geq 4$ geodesics contained in $A^{(1)}$ and passing through v_1. Then the argument in the proof of Lemma 7.7 again shows that their images in Δ_2 have a common point. Since the images of the two geodesics containing the two edges of C_i incident to v_1 intersect at w_{C_i}, we have $w_{C_1} = w_{C_2}$. Together with the previous paragraph we conclude $w_{C_1} = w_{C_2}$ for any two adjacent chambers $C_1, C_2 \subset A$. Since A is chamber connected, $w_{C_1} = w_{C_2}$ for any two chambers C_1, C_2 in A. It follows that there is a vertex $w \in \Delta_2$ such that $\xi'\eta'$ contains w for any geodesic $\xi\eta$ contained in the 1-skeleton of A.

Since the map $h : \partial\Delta_1 \to \partial\Delta_2$ is a homeomorphism, it induces a homeomorphism $h \times h : \partial\Delta_1 \times \partial\Delta_1 \to \partial\Delta_2 \times \partial\Delta_2$ satisfying $h \times h(\partial\Delta_1 \times \partial\Delta_1 - \{(\xi, \xi) : \xi \in \partial\Delta_1\}) = \partial\Delta_2 \times \partial\Delta_2 - \{(\xi, \xi) : \xi \in \partial\Delta_2\}$. Let $E_1 = \{(\xi, \eta) : \xi, \eta \in \partial A \text{ and } \xi\eta \subset A^{(1)}\}$ and $E_2 = \{(\xi', \eta') : \xi', \eta' \in \partial\Delta_2 \text{ and } \xi'\eta' \ni w\}$. Then $h \times h(E_1) \subset E_2$. On the other hand, $\overline{E_2} \subset \partial\Delta_2 \times \partial\Delta_2 - \{(\xi, \xi) : \xi \in \partial\Delta_2\}$ and $\overline{E_1} \not\subset \partial\Delta_1 \times \partial\Delta_1 - \{(\xi, \xi) : \xi \in \partial\Delta_1\}$, contradicting to the fact that $h \times h$ is a homeomorphism.

□

8. **The right triangle case.** In this section we prove Theorem 1.3 when both R_1 and R_2 are right triangles but none of them is $(2,3,8)$. The exceptional case $(2,3,8)$ will be considered in the next section. We remind the reader that Proposition 5.2 shall be used in the manner indicated in Remark 7.6.

8.1. *Types of geodesics.* Let Δ be a Fuchsian building. Each complete geodesic contained in $\Delta^{(1)}$ is a labeled complex with the induced labeling. Two complete geodesics contained in $\Delta^{(1)}$ have the *same type* if they are isomorphic as labeled complexes. Let us first find all possible types of geodesics. Since geodesics are contained in apartments and all apartments are isomorphic as labeled complexes, we only need to look at one apartment. Let A be a fixed apartment and $C \subset A$ a fixed chamber. Since the action of Coxeter group preserves the labeling, for each complete geodesic $c_1 \subset A^{(1)}$, there is a complete geodesic $c_2 \subset A^{(1)}$ containing an edge of C such that c_1 and c_2 are of the same type. So there are at most k different types of geodesics. Let $c \subset A^{(1)}$ be a complete geodesic and $v \in c$ a vertex labeled by $\{i, i+1\}$. Note that if $m(v)$ is even, then the two edges of c incident to v have the same labeling; and if $m(v) = 3$, then these two edges are labeled by $\{i\}$ and $\{i+1\}$ respectively. From this we obtain a description of all different types of geodesics in Δ. If $m(v) = 3$ for all vertices, then there is only one type of geodesics and the edges on each complete geodesic are periodically labeled by $\{1\}, \{2\}, \cdots, \{k\}$. Assume there is at least one vertex v with $m(v) \neq 3$. We consider the boundary ∂R of R and let $\partial R'$ be the complement in ∂R of all vertices v with $m(v) \neq 3$. Let L be a component of $\partial R'$. Then the edges of L are linearly labeled by $\{i\}, \{i+1\}, \cdots, \{j\}$. Let c be a complete geodesic containing an edge labeled by some $\{l\}$ satisfying $i \leq l \leq j$. Then the edges of c are periodically labeled by $\{i\}, \{i+1\}, \cdots, \{j\}, \{j\}, \cdots, \{i+1\}, \{i\}$. It is clear that this establishes a 1-to-1 correspondence between the types of geodesics and the components of $\partial R'$.

Suppose $\partial R'$ has at least two components, or equivalently, there are at least two vertices w of R with $m(w) \neq 3$. Now let $v \in \Delta$ be a fixed vertex. Consider the set $\mathcal{D}_v$ of all complete geodesics through v that are contained in $\Delta^{(1)}$. The above 1-to-1 correspondence between the types of geodesics and the components of $\partial R'$ shows the following: if $m(v) = 3$, then all the geodesics in $\mathcal{D}_v$ have the same type; if $m(v) \neq 3$, then $c_1, c_2 \in \mathcal{D}_v$ have the same type if and only if they make an even angle at v. In general, if two geodesics $c_1, c_2 \subset \Delta^{(1)}$ intersect at a vertex v, they actually make 4 angles at v. If $m(v)$ is even, then either all 4 angles are even, or all 4 angles are odd; if $m(v) = 3$, then 2 of the angles are even and two are odd. Hence two intersecting geodesics $c_1, c_2 \subset \Delta^{(1)}$ have the same type if and only if at least one of the angles they make is even.

Now suppose R is a right triangle. If $R \neq (2,3,8)$, then there are three different types of geodesics; if $R = (2,3,8)$, then there are only two different types of geodesics.

8.2. *Two disjoint geodesics at different sides.* Recall that there are only 6 right triangles (up to isometry) that can appear as the chamber of a Fuchsian building: $(2,8,8)$, $(2,6,8)$, $(2,6,6)$, $(2,4,6)$, $(2,4,8)$, $(2,3,8)$.

In Lemma 6.6 and Propositions 6.7 and 6.8 we have done some preliminary studies on disjoint geodesics that are at different sides. Here we continue this analysis.

Lemma 8.1. *Suppose $R \neq (2,3,8)$ is a right triangle. Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ (ξ_1, ξ_2, η_1, $\eta_2 \in \partial\Delta$) be geodesics that are disjoint and are at different sides. Let $x_i \in \xi_1\xi_2$, $y_i \in \eta_1\eta_2$ be vertices such that $x_i\eta_i \cap \eta_1\eta_2$, $y_i\xi_i \cap \xi_1\xi_2$ are rays and $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$, $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. If $y_1' = y_2'$ and $y_1 \neq y_2$, then $R = (2,4,6)$ or $(2,4,8)$, $m(y_1') = 4$, $m(y_1) = m(y_2) = 6$ or 8, and there is a configuration as shown in Figure 4.*

Proof. Recall in a generalized m-gon opposite vertices have the same type if m is even. Since $\mathrm{Link}(\Delta, y_1')$ is a generalized $m(y_1')$-gon and $m(y_1')$ is even, it follows from the assumptions that the angle $\angle_{y_1'}(y_1, y_2)$ is even. Proposition 5.2 applied to (y_1, y_2, y_1') shows that y_1y_2 contains exactly one vertex in the interior, $m(y_1) = m(y_2) \geq 4$ and $\angle_{y_1}(y_1', y_2) = \angle_{y_2}(y_1', y_1) = \frac{\pi}{m(y_1)}$.

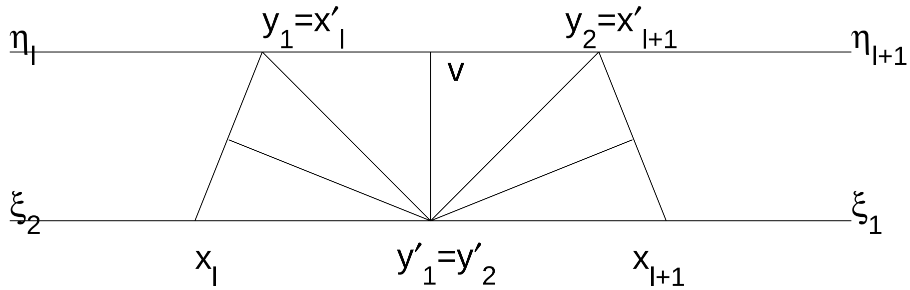

FIGURE 4. Two disjoint geodesics at different sides

First we assume one of x'_1, x'_2 does not lie on y_1y_2. Then there are i and j with $y_{j+1} \in \text{interior}(y_jx'_i)$. Here, below and in Figure 4 the indices $j+1, i+1, l+1$ are taken mod 2. Notice $\angle_{y_{j+1}}(y'_1, x'_i) \geq \pi - \angle_{y_{j+1}}(y_j, y'_1) \geq 3\pi/4$. If $x_i \in y'_1\xi_{j+1}$, then $x_iy_{j+1} = x_iy'_1 \cup y'_1y_{j+1}$ and (x_i, y_{j+1}, x'_i) is a triangle with sum of angles $\geq \angle_{y_{j+1}}(y'_1, x'_i) + \pi/8 + \pi/8 = \pi$, which is impossible. Hence $x_i \in y'_1\xi_j$. In this case $x_iy_j = x_iy'_1 \cup y'_1y_j$ and (x_i, y_j, x'_i) is a triangle. By the definition of x_i, $\angle_{x'_i}(x_i, y_j)$ either $= \pi$ or is an even angle. $\angle_{x'_i}(x_i, y_j) = \pi$ is impossible since otherwise $x_ix'_i \cup x'_iy_j$ and $x_iy'_1 \cup y'_1y_j$ would be two distinct geodesic segments from x_i to y_j. Therefore (x_i, y_j, x'_i) has an even angle at x'_i. Since $y_{j+1} \in \text{interior}(y_jx'_i)$, Proposition 5.2 implies $m(y_{j+1}) = 2$, contradicting to the above observation that $m(y_1) = m(y_2) \geq 4$. The contradiction shows $x'_1, x'_2 \in y_1y_2$.

Assume $x'_i \in \text{interior}(y_1y_2)$ for some $i = 1, 2$. Then x'_i is the only vertex in the interior of y_1y_2. There is some $j = 1, 2$ with $x_i \in y'_1\xi_j$. The uniqueness of the geodesic x_iy_j shows $\angle_{x'_i}(x_i, y_j) \neq \pi$. Hence $\angle_{x'_i}(x_i, y_j)$ is an even angle. Since x'_iy_j contains no vertex in the interior, Proposition 5.2 applied to (x_i, x'_i, y_j) implies that $m(y'_1) = 2$, contradicting to the fact that (y'_1, y_1, y_2) has an even angle at y'_1. Therefore we conclude $\{x'_1, x'_2\} \subset \{y_1, y_2\}$.

There is some $i = 1, 2$ with $x'_1 = y_i$. The uniqueness of geodesic implies $x_1 \in y'_1\xi_{i+1}$ and $\angle_{x'_1}(x_1, y_{i+1}) \neq \pi$. We have $x_1y_{i+1} = x_1y'_1 \cup y'_1y_{i+1}$ and (x'_1, x_1, y_{i+1}) has an even angle $\angle_{x'_1}(x_1, y_{i+1})$. Since y_iy_{i+1} has a vertex v in the interior, Proposition 5.2 implies the following: $R = (2, 4, 6)$ or $(2, 4, 8)$; $m(v) = 2$; $m(y'_1) = 4$; there is a vertex $v_1 \in \text{interior}(x_1x'_1)$ with $m(v_1) = 2$; (y'_1, x_1, v_1), (y'_1, x'_1, v_1), (y'_1, x'_1, v), (y'_1, y_{i+1}, v) are chambers and $S(T)$ is the union of these 4 chambers. Applying the same argument to x'_2 we obtain the following: $x'_2 = y_{i+1}$, $x_2 \in \text{interior}(y'_1\xi_i)$; there is a vertex $v_2 \in \text{interior}(x_2x'_2)$ such that (y'_1, x_2, v_2) and (y'_1, x'_2, v_2) are chambers.

□

Lemma 8.2. *Suppose $R \neq (2, 3, 8)$ is a right triangle. Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ (ξ_1, ξ_2, η_1, $\eta_2 \in \partial\Delta$) be geodesics that are disjoint and are at different sides. Let $x_i \in \xi_1\xi_2$, $y_i \in \eta_1\eta_2$ be vertices such that $x_i\eta_i \cap \eta_1\eta_2$, $y_i\xi_i \cap \xi_1\xi_2$ are rays and $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$, $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. Suppose $y'_1 \in$ interior$(x_1\xi_1)$ and $y_1 \in$ interior$(x'_1\eta_1)$. Then $R = (2, 4, 6)$ or $(2, 4, 8)$, and there must be a configuration isomorphic to the one in Figure 4.*

Proof. If $x'_2 = x'_1$, then we are done by Lemma 8.1. We suppose $x'_2 \neq x'_1$. By Proposition 6.7, $x'_2 \in \text{interior}(x'_1\eta_2)$. Note $m(x'_1) \neq 2$, otherwise $\angle_{x'_1}(x_1, y_1) = \angle_{x'_1}(x_1, x'_2) = \pi$, contradicting to Proposition 6.7. First assume $x_2 \in y'_1\xi_1$. In this case $x_2y_1 = x_2y'_1 \cup y'_1y_1$ and (x_2, y_1, x'_2) has an even angle $\angle_{x'_2}(x_2, y_1)$. Since $y_1x'_2$ contains the vertex x'_1 in the interior, Proposition 5.2 implies $m(x'_1) = 2$, a contradiction.

Note Proposition 6.7 implies $x_2 \neq x_1$. Next we assume $x_2 \in \text{interior}(x_1y'_1)$. Since (y'_1, x_1, y_1) has an even angle at y'_1, Proposition 5.2 implies $m(x_2) = 2$ and $\angle_{x_2}(x_1, x'_1) = \pi/2$. If $\angle_{x_2}(x_1, x'_2) = \pi$, then $\angle_{x_2}(y'_1, x'_2) = \pi$, contradicting to Proposition 6.7. Hence $\angle_{x_2}(x_1, x'_2) = \pi/2$, which implies

$\angle_{x_2}(x_2', x_1') = \pi$ and so $x_1'x_2' = x_1'x_2 \cup x_2x_2'$ is not contained in $\eta_1\eta_2$, a contradiction. We conclude $x_2 \in \text{interior}(x_1\xi_2)$.

Since (y_1', y_1, x_1) is a triangle with an even angle $\angle_{y_1'}(x_1, y_1)$ and the side x_1y_1 contains the vertex x_1' with $m(x_1') \neq 2$ in the interior, Proposition 5.2 implies $R = (2,4,6)$ or $(2,4,8)$, $m(x_1') = 4$ and $\angle_{x_1}(x_1', y_1') = \pi/6$ or $\pi/8$. It follows that $\angle_{x_1}(x_2, x_1') = 5\pi/6$ or $7\pi/8$. Consider the quadrilateral $Q = (x_1, x_1', x_2', x_2)$: $\angle_{x_1'}(x_1, x_2') \geq \pi/2$ since it is even and $m(x_1') = 4$; $\angle_{x_2'}(x_2, x_1') \geq \pi/4$ since it is even. Proposition 5.3 implies $n(Q) \leq 2$, which is impossible since there are at least 5 or 7 chambers in $S(Q)$ incident to x_1.

□

Lemma 8.3. *Suppose $R \neq (2,3,8)$ is a right triangle. Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ (ξ_1, ξ_2, η_1, $\eta_2 \in \partial\Delta$) be geodesics that are disjoint and are at different sides. Let $x_i \in \xi_1\xi_2$, $y_i \in \eta_1\eta_2$ be vertices such that $x_i\eta_i \cap \eta_1\eta_2$, $y_i\xi_i \cap \xi_1\xi_2$ are rays and $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$, $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. Suppose $y_1' \in$ interior$(x_1\xi_1)$ and $y_1 = x_1'$. Then $R = (2,4,6)$ or $(2,4,8)$, and there must be a configuration isomorphic to the one in Figure 4.*

Proof. The lemma follows from Lemma 8.1 if $y_2' = y_1'$. We assume $y_2' \neq y_1'$. Proposition 6.7 implies $y_2' \in \text{interior}(y_1'\xi_2)$. We first assume $y_2' \in \text{interior}(y_1'x_1)$. Note (y_1', x_1, y_1) is a triangle with an even angle at y_1' and the side $y_1'x_1$ contains the vertex y_2' in the interior. Proposition 5.2 implies $m(y_2') = 2$, which in turn implies $\angle_{y_2'}(y_1', y_2) = \angle_{y_2'}(x_1, y_2) = \pi$, contradicting to Proposition 6.7. Hence $y_2' \in x_1\xi_2$.

Now assume $y_2' = x_1$. The uniqueness of geodesic implies that $y_2 \in y_1\eta_2$. Proposition 6.7 implies $y_2 \neq y_1$. The segment x_1y_1 is the side of (y_1', x_1, y_1) opposite to an even angle. Proposition 5.2 implies x_1y_1 contains exactly one vertex in the interior. On the other hand, (x_1, y_1, y_2) has an even angle at y_1. By Proposition 5.2, $R = (2,4,6)$ or $(2,4,8)$, y_1y_2 contains exactly one vertex v in the interior and $m(v) = 2$, and x_1y_2 also contains exactly one vertex v' in the interior and $m(v') = 4$. Now let $x_2 \in \xi_1\xi_2$ be a vertex such that $x_2\eta_2 \cap \eta_1\eta_2 = x_2'\eta_2$ is a ray and $x_2\eta_2 \cap \xi_1\xi_2 = \{x_2\}$. By Proposition 6.7 and Lemma 8.1 we may assume $x_2' \in \text{interior}(y_1\eta_2)$. The argument in the first paragraph shows $x_2' \neq v$. Assume $x_2' = y_2$. The uniqueness of geodesic shows $x_2 \in \text{interior}(x_1\xi_1)$. The triangle (x_1, x_2, y_2) has an even angle at x_1 and the side x_1y_2 contains the vertex v' with $m(v') = 4$, contradicting to Proposition 5.2. Therefore we must have $x_2' \in \text{interior}(y_2\eta_2)$. The lemma follows from Lemma 8.2 if $x_2 \in \text{interior}(x_1\xi_2)$. Proposition 6.7 implies $x_2 \neq x_1$. Therefore $x_2 \in y_1'\xi_1$. In this case $x_2y_1 = x_2y_1' \cup y_1'y_1$, (x_2, y_1, x_2') has an even angle at x_2' and the side y_1x_2' contains two vertices v, y_2 in the interior, contradicting to Proposition 5.2.

Finally we assume $y_2' \in \text{interior}(x_1\xi_2)$. Proposition 6.7 implies $y_2 \neq y_1$. The lemma follows from Lemma 8.2 if $y_2 \in \text{interior}(y_1\eta_1)$. We now assume $y_2 \in \text{interior}(y_1\eta_2)$. Then we have a quadrilateral $Q = (x_1, y_1, y_2, y_2')$. We shall show such a quadrilateral does not exist. We only give the proof for $R = (2,4,6)$. The other cases can be handled similarly. Recall (y_1', x_1, y_1) has an even angle at y_1'. Proposition 5.2 implies that $m(x_1) = m(y_1)$ and $\angle_{y_1}(x_1, y_1') = \angle_{x_1}(y_1, y_1') = \pi/4$ or $\pi/6$. The quadrilateral Q has even angles at y_1 and y_2'. Notice $\angle_{x_1}(y_1, y_2') \geq \pi - \angle_{x_1}(y_1, y_1')$ and $\angle_{y_1}(x_1, y_2) \geq 2\angle_{y_1}(x_1, y_1') = 2\angle_{x_1}(y_1, y_1')$. If $\angle_{x_1}(y_1, y_1') = \pi/6$, then $\angle_{x_1}(y_1, y_2') = 5\pi/6$ and Proposition 5.3 implies $n(Q) \leq 4$, which is impossible because there are at least 5 chambers in $S(Q)$ incident to x_1. If $\angle_{x_1}(y_1, y_1') = \pi/4$, then $\angle_{x_1}(y_1, y_2') = 3\pi/4$ and Proposition 5.3 implies $n(Q) \leq 3$. It follows that $S(Q)$ is the union of the 3 chambers in $S(Q)$ incident to x_1. But such a union has only one even angle, contradiction.

□

Proposition 8.4. *Suppose $R \neq (2,3,8)$ is a right triangle. Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$) be geodesics that are disjoint and are at different sides. Then $R = (2,4,6)$ or $(2,4,8)$ and there must be a configuration isomorphic to the one in Figure 4. In particular, $\xi_1\xi_2$ and $\eta_1\eta_2$ are of different types.*

Proof. The proof is similar to that of Proposition 6.9. Lemma 6.6 implies there are vertices $x_i \in \xi_1\xi_2$, $y_i \in \eta_1\eta_2$ such that $x_i\eta_i \cap \eta_1\eta_2$, $y_i\xi_i \cap \xi_1\xi_2$ are rays and $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$, $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. By Lemmas 8.1, 8.2 and 8.3, $R = (2,4,6)$ or $(2,4,8)$, and there is a configuration isomorphic to the one in Figure 4 if any one of the following occurs: Proposition 6.7 (2), Proposition 6.8 (1), (2). So we can assume only Proposition 6.7 (1) and Proposition 6.8 (3) can occur. Then the proof of Proposition 6.9 shows that there is a quadrilateral Q with one side xy such that the angles at x and y are even and xy contains two vertices in the interior. Corollary 5.5 implies $R = (2,3,8)$, contradicting to our assumption.

□

Let us make some observations about the configuration in Figure 4. Notice $\angle_v(\eta_1, y_1') = \angle_v(\eta_2, y_1') = \pi/2$ and $\angle_{y_1'}(\xi_1, v) = \angle_{y_1'}(\xi_2, v) = 3\pi/4$. It follows that vy_1' is the shortest geodesic between $\xi_1\xi_2$ and $\eta_1\eta_2$. Consider the quadrilateral $Q = (x_l, x_l', x_{l+1}', x_{l+1})$. Notice Q is uniquely determined by $\xi_1\xi_2$ and the segment vy_1': the geodesic segment in $\mathrm{Link}(\Delta, y_1')$ from the initial direction of $y_1'v$ to the initial direction of $y_1'\xi_1$ consists of 3 edges, and these 3 edges determine 3 chambers incident to y_1'; similarly the geodesic segment in $\mathrm{Link}(\Delta, y_1')$ from the initial direction of $y_1'v$ to the initial direction of $y_1'\xi_2$ determine 3 chambers; the union of these 6 chambers is a disk whose boundary is Q. We denote this Q by $Q(\xi_1\xi_2, \eta_1\eta_2)$.

8.3. *Geodesics in the same apartment.* The eventual goal of this section is to show

Proposition 8.5. *Let Δ_1, Δ_2 be two Fuchsian buildings, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. Let A be an apartment in Δ_1 and $v \in A$ a vertex. If both R_1 and R_2 are right triangles different from $(2,3,8)$, then the geodesics in $\mathcal{D}'_{A,v}$ intersect in a unique vertex of Δ_2.*

One first has to show that any two geodesics in $\mathcal{D}'_{A,v}$ have nonempty intersection (Proposition 8.13).

Lemma 8.6. *Suppose $R \neq (2,3,8)$ is a right triangle. Let $c, c_1, c_2 \subset \Delta^{(1)}$ be complete geodesics such that any two of them are at different sides. If c_1 and c_2 have the same type and $c \cap c_1 = \phi$, then $c \cap c_2 = \phi$.*

Proof. Assume $c \cap c_2 \neq \phi$. Since $c \cap c_1 = \phi$, Proposition 8.4 implies $R = (2,4,6)$ or $(2,4,8)$ and there is a configuration as shown in Figure 4, where either $c_1 = \xi_1\xi_2$, $c = \eta_1\eta_2$ or $c = \xi_1\xi_2$, $c_1 = \eta_1\eta_2$. Proposition 8.4 also implies $c_1 \cap c_2 \neq \phi$ because c_1 and c_2 have the same type and are at different sides. Let $p = c_2 \cap \eta_1\eta_2$, $q = c_2 \cap \xi_1\xi_2$. Then $pq \subset c_2 \subset \Delta^{(1)}$. We will show $p \in v\eta_{l+1}$ is impossible. Since the configuration is symmetric about vy_1', $p \in v\eta_l$ is also impossible, and we have a contradiction. We shall only write down the proof for $R = (2,4,6)$, the case $R = (2,4,8)$ can be similarly handled.

First assume $p \in \mathrm{interior}(y_2\eta_{l+1})$. Then $q \in \mathrm{interior}(x_{l+1}\xi_1)$ is impossible, since otherwise $px_{l+1} = py_2 \cup y_2x_{l+1}$ and (p, x_{l+1}, q) has an angle $\angle_{x_{l+1}}(p,q) = \pi - \angle_{x_{l+1}}(y_2, y_1') = 5\pi/6$, which is impossible. Note $q = x_{l+1}$ cannot occur since otherwise $c_2 \cap \eta_1\eta_2$ contains a nontrivial segment py_2. Note $\angle_{y_1'}(y_2, x_l) = \pi$. If $q \in y_1'\xi_2$, then $y_2q = y_2y_1' \cup y_1'q$ and (p, y_2, q) has an angle $\angle_{y_2}(p,q) \geq \angle_{y_2}(p, x_{l+1}) - \angle_{y_2}(x_{l+1}, y_1') = \pi - \pi/6 = 5\pi/6$, a contradiction.

Now assume $p = y_2$. The angle $\angle_{x_{l+1}}(y_2, \xi_1) = 5\pi/6$ implies $q \notin \mathrm{interior}(x_{l+1}\xi_1)$. Note that if two intersecting geodesics are at different sides, then none of the 4 angles they make can be π. It follows that $q \neq x_{l+1}, y_1'$ since $\angle_{y_2}(x_{l+1}, \eta_{l+1}) = \angle_{y_1'}(y_2, \xi_2) = \pi$. The fact $y_1' \in y_2\xi_2$ implies $q \notin \mathrm{interior}(y_1'\xi_2)$ since $c_2 \cap \xi_1\xi_2$ is a single point.

Finally we assume $p = v$. Note $m(v) = 2$ and $m(y_1') = 4$. If $q \neq y_1'$, then (v, y_1', q) is a triangle with $\angle_v(y_1', q) = \pi/2$ and $\angle_{y_1'}(v, q) = 3\pi/4$, which is impossible. Hence $q = y_1'$. It follows that the type of $c_2 \supset vy_1'$ is different from the types of $\xi_1\xi_2$ and $\eta_1\eta_2$, contradicting to the assumption.

□

Lemma 8.7. *Let Δ be a Fuchsian building with chamber $R \neq (2,3,8)$, $v_0 \in \Delta$ a vertex and $\xi_1, \xi_2, \eta_1, \eta_2, \eta_3 \in \partial\Delta$. Suppose the following conditions are satisfied:*
(1) $\xi_1\xi_2 \subset \Delta^{(1)}$;
(2) $\eta_i\eta_j = v_0\eta_i \cup v_0\eta_j \subset \Delta^{(1)}$ for all $1 \leq i, j \leq 3$, $i \neq j$;
(3) $\xi_1\xi_2$ and $\eta_i\eta_j$ are at different sides for all $1 \leq i, j \leq 3$, $i \neq j$.
Then $v_0 \in \xi_1\xi_2$.

Proof. First assume $\xi_1\xi_2 \cap \eta_i\eta_j \neq \phi$ for some $i \neq j$, say, $\xi_1\xi_2 \cap \eta_1\eta_2 \neq \phi$. Lemma 6.12 implies $\xi_1\xi_2 \cap \eta_1\eta_2$ is some vertex $v' \in \Delta$. If $v' = v_0$, then we are done. Suppose $v' \neq v_0$. We may assume $v' \in \text{interior}(v_0\eta_1)$. Since $\eta_1\eta_i = v_0\eta_1 \cup v_0\eta_i$ for $i = 2, 3$, Lemma 6.12 also implies $\xi_1\xi_2 \cap v_0\eta_i = \phi$ for $i = 2, 3$. It follows that $\xi_1\xi_2 \cap \eta_2\eta_3 = \phi$. Now $\xi_1\xi_2$ and $\eta_2\eta_3$ are disjoint and are at different sides, and $v'\eta_i \cap \eta_2\eta_3 = v_0\eta_i$ $(i = 2, 3)$ with $v' \in \xi_1\xi_2$, contradicting to Proposition 6.7.

Now we assume $\xi_1\xi_2 \cap \eta_i\eta_j = \phi$ for all $i \neq j$. Proposition 8.4 applied to $\xi_1\xi_2$ and $\eta_1\eta_2$ shows that $R = (2,4,6)$ or $(2,4,8)$, and there is a configuration isomorphic to the one in Figure 4. Consider the perpendicular xy $(x \in \eta_1\eta_2,\ y \in \xi_1\xi_2)$ between $\eta_1\eta_2$ and $\xi_1\xi_2$. We claim $v_0 = x$. Suppose $v_0 \neq x$, say $v_0 \in \text{interior}(x\eta_2)$. Since $\eta_1\eta_i = v_0\eta_1 \cup v_0\eta_i$ for $i = 2, 3$ and Δ is CAT(-1), $d(z, \xi_1\xi_2) > d(x, y)$ holds for all $z \in v_0\eta_i$, $i = 2, 3$. It follows that $d(\eta_2\eta_3, \xi_1\xi_2) > d(x, y)$. Here we have a contradiction since Proposition 8.4 applied to $\xi_1\xi_2$ and $\eta_2\eta_3$ shows that $d(\eta_2\eta_3, \xi_1\xi_2) = d(x, y)$. Hence $v_0 = x$. Notice xy is the perpendicular between $\xi_1\xi_2$ and $\eta_i\eta_j$ for all $i \neq j$. By Proposition 8.4 $m(v_0) = 2$ or 4. We consider these two cases separately.

First assume $m(v_0) = 2$. Then $m(y) = 4$. Consider the quadrilaterals $Q(\xi_1\xi_2, \eta_i\eta_j)$, $i \neq j$. By the observation made after Proposition 8.4, $Q(\xi_1\xi_2, \eta_i\eta_j)$ is uniquely determined by $\xi_1\xi_2$ and yx. It follows that $Q(\xi_1\xi_2, \eta_1\eta_2) = Q(\xi_1\xi_2, \eta_1\eta_3)$, which is not true since $Q(\xi_1\xi_2, \eta_1\eta_3)$ contains an edge from $v_0\eta_3$ while $Q(\xi_1\xi_2, \eta_1\eta_2)$ does not.

Now assume $m(v_0) = 4$. Then $m(y) = 2$. Let $y_1, y_2 \in \xi_1\xi_2$ be the two vertices on $\xi_1\xi_2$ adjacent to y, and $\sigma, \sigma_i \in \text{Link}(\Delta, v_0)$ $(i = 1, 2)$ the initial directions of v_0y, v_0y_i respectively. Also let $\omega_j \in \text{Link}(\Delta, v_0)$ be the initial direction of $v_0\eta_j$. Proposition 8.4 implies that for each j, $1 \leq j \leq 3$, the segment $\sigma\omega_j \subset \text{Link}(\Delta, v_0)$ consists of three edges and the initial edge is $\sigma\sigma_1$ or $\sigma\sigma_2$. Hence there is a map $f : \{\eta_1, \eta_2, \eta_3\} \to \{\sigma\sigma_1, \sigma\sigma_2\}$, where $f(\eta_j)$ is the initial edge of the segment $\sigma\omega_j$. Proposition 8.4 also implies that f is injective, which is clearly not true since $\{\sigma\sigma_1, \sigma\sigma_2\}$ contains only two elements while $\{\eta_1, \eta_2, \eta_3\}$ contains three elements.

□

Lemma 8.8. *Suppose $R_1, R_2 \neq (2,3,8)$ are right triangles. Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta_1^{(1)}$ $(\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta_1)$ be two geodesics locally contained in an apartment. Let $v = \xi_1\xi_2 \cap \eta_1\eta_2$. If $m(v) = 2$, then $\xi_1'\xi_2' \cap \eta_1'\eta_2' \neq \phi$.*

Proof. Pick an edge $e = vv'$ such that $v'\eta_i = v'v \cup v\eta_i$ $(i = 1, 2)$. We extend the edge vv' to obtain a ray $v\eta_3$, where $\eta_3 \in \partial\Delta_1$. By construction, $\eta_i\eta_j = v\eta_i \cup v\eta_j \subset \Delta_1^{(1)}$. It follows that $\eta_i'\eta_j' \subset \Delta_2^{(1)}$. Since $R_2 \neq (2,3,8)$, Lemma 7.5 implies there is a vertex $w \in \Delta_2$ such that $\eta_i'\eta_j' = w\eta_i' \cup w\eta_j'$. On the other hand, since $m(v) = 2$, $\xi_1\xi_2$ and $\eta_i\eta_j$ make a right angle and are locally contained in an apartment. Lemma 6.2 implies that $\xi_1\xi_2$ and $\eta_i\eta_j$ are at different sides. Hence $\xi_1'\xi_2'$ and $\eta_i'\eta_j'$ are also at different sides. Now Lemma 8.7 implies $w \in \xi_1'\xi_2'$. Consequently, $w \in \xi_1'\xi_2' \cap \eta_1'\eta_2'$.

□

Lemma 8.9. *Suppose R_1 and R_2 are right triangles. Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta_1^{(1)}$ be two geodesics that have nonempty intersection. If at least one of the angles that $\xi_1\xi_2$ and $\eta_1\eta_2$ make is even, then $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ have the same type.*

Proof. Let $v \in \xi_1\xi_2 \cap \eta_1\eta_2$ be a vertex. We may assume $\angle_v(\xi_1, \eta_1)$ is even. Since $\text{Link}(\Delta_1, v)$ is a thick spherical building, Proposition 2.4 implies there is an edge e incident to v such that e makes an angle π with both $v\xi_1$ and $v\eta_1$. We extend e to obtain a geodesic ray $v\omega$. Then $\omega\xi_1 = v\omega \cup v\xi_1$ and

$\omega\eta_1 = v\omega \cup v\eta_1$. Consider the geodesics $\xi_1'\xi_2'$, $\eta_1'\eta_2'$, $\omega'\xi_1'$ and $\omega'\eta_1'$ in Δ_2. We note if two geodesics in Δ_2 intersect in a ray, then they have the same type. Now $\xi_1'\xi_2' \cap \omega'\xi_1'$, $\omega'\xi_1' \cap \omega'\eta_1'$, $\omega'\eta_1' \cap \eta_1'\eta_2'$ are all rays. So the 4 geodesics have the same type. In particular, $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ have the same type.

□

Corollary 8.10. *Suppose R_1, R_2 are right triangles. Let $\xi_1\xi_2, \eta_1\eta_2, \zeta_1\zeta_2 \subset \Delta_1^{(1)}$ be three geodesics such that any two of $\xi_1'\xi_2', \eta_1'\eta_2', \zeta_1'\zeta_2'$ have nonempty intersection. If $\xi_1\xi_2, \eta_1\eta_2, \zeta_1\zeta_2$ pairwise have different types, then $\xi_1'\xi_2' \cap \eta_1'\eta_2' \cap \zeta_1'\zeta_2' = \phi$.*

Proof. Note that there are at most two types of geodesics through a fixed vertex. Suppose $\xi_1'\xi_2' \cap \eta_1'\eta_2' \cap \zeta_1'\zeta_2'$ contains a vertex w. Then two of $\xi_1'\xi_2'$, $\eta_1'\eta_2'$, $\zeta_1'\zeta_2'$, say, $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ have the same type. It follows that at least one of the angles that $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ make is even. Lemma 8.9 then implies $\xi_1\xi_2$ and $\eta_1\eta_2$ have the same type, contradicting to the assumption.

□

Lemma 8.11. *Let Δ be a Fuchsian building with $R = (2, 4, 6)$ or $(2, 4, 8)$. Suppose $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$) are disjoint and are at different sides. Assume $\eta_1\eta_2$ contains vertices v with $m(v) = 2$. Let pq ($p \in \eta_1\eta_2$, $q \in \xi_1\xi_2$) be the perpendicular between $\eta_1\eta_2$ and $\xi_1\xi_2$. If $\xi_3\xi_4 \subset \Delta^{(1)}$ ($\xi_3, \xi_4 \in \partial\Delta$) is a geodesic such that $\xi_3\xi_4 \cap \xi_1\xi_2 \neq \phi$, $\xi_3\xi_4 \cap \eta_1\eta_2 = \phi$, and that $\xi_3\xi_4$ and $\eta_1\eta_2$ are at different sides, then $q \in \xi_3\xi_4$ and pq is also the perpendicular between $\eta_1\eta_2$ and $\xi_3\xi_4$.*

Proof. We shall only write down the proof for $R = (2, 4, 8)$, the case for $R = (2, 4, 6)$ is similar. Suppose $q \notin \xi_3\xi_4$. We may assume $\xi_3\xi_4 \cap \xi_1\xi_2 = xy \subset \text{interior}(q\xi_2)$ with $x \in \text{interior}(q\xi_2)$ and $y \in x\xi_2 \cup \{\xi_2\}$. Let $p'q'$ ($p' \in \eta_1\eta_2$, $q' \in \xi_3\xi_4$) be the perpendicular between $\eta_1\eta_2$ and $\xi_3\xi_4$. By Proposition 8.4 and our assumption, $m(p) = m(p') = 2$ and $\angle_{q'}(p', \xi_j) = 3\pi/4$ ($j = 3, 4$). Note $\angle_{y_2}(\eta_1, q) = \angle_{x_2}(y_1, \xi_2) = 7\pi/8$. If $p' \in \text{interior}(y_2\eta_1)$, then $Q = (q', p', y_2, x)$ has angle sum $\geq 3\pi/4 + \pi/2 + 7\pi/8 + \pi/8 > 2\pi$, impossible. If $p' = p$, then $q'q = q'p' \cup pq$ since $m(p) = 2$; hence $T = (q', q, x)$ has angle sum $\geq 3\pi/4 + 3\pi/4 + \pi/8 > \pi$, again impossible. Now assume $p' \in \text{interior}(y_1\eta_2)$. If $x \in \text{interior}(x_2\xi_2)$, then (q', p', x_2, x) has angle sum $\geq 3\pi/4 + \pi/2 + 7\pi/8 + \pi/8 > 2\pi$, impossible. If $x = x_2$, then (q', p', x_2) has angle sum $\geq 3\pi/4 + \pi/2 + \pi/8 > \pi$, also impossible. Hence $q \in \xi_3\xi_4$. Proposition 8.4 implies $d(\eta_1\eta_2, \xi_3\xi_4) = d(\eta_1\eta_2, \xi_1\xi_2) = d(p, q)$. So pq realizes the distance between $\eta_1\eta_2$ and $\xi_3\xi_4$. Since Δ is CAT(-1), pq must be the unique perpendicular between $\eta_1\eta_2$ and $\xi_3\xi_4$.

□

Lemma 8.12. *Let Δ be a Fuchsian building with $R = (2, 4, 6)$ or $(2, 4, 8)$. Suppose $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ ($\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta$) are disjoint and are at different sides. Assume $\eta_1\eta_2$ contains vertices v with $m(v) = 2$. Let pq ($p \in \eta_1\eta_2$, $q \in \xi_1\xi_2$) be the perpendicular between $\eta_1\eta_2$ and $\xi_1\xi_2$, and $p_1, p_2 \in \eta_1\eta_2$ the two vertices on $\eta_1\eta_2$ adjacent to p. If $\eta_3\eta_4 \subset \Delta^{(1)}$ ($\eta_3, \eta_4 \in \partial\Delta$) is a geodesic such that $\eta_3\eta_4 \cap \eta_1\eta_2 = v$ is a vertex, $\eta_3\eta_4 \cap \xi_1\xi_2 = \phi$, and $\eta_3\eta_4$ and $\xi_1\xi_2$ are at different sides, then $v = p_1$ or p_2. Furthermore, if $p'q'$ ($p' \in \eta_3\eta_4$, $q' \in \xi_1\xi_2$) is the perpendicular between $\eta_3\eta_4$ and $\xi_1\xi_2$, then $q' = q$ and (v, q, p') is the boundary of a chamber.*

Proof. An argument similar to the proof of Lemma 8.11 shows that $q' = q$ and $v = p_1$ or p_2. Let $Q = (p, q, p', v)$. Q has right angles at p and p' and even angles at q and v: $\angle_q(p, p')$ is even because $m(q) = 4$ and $\angle_q(p, \xi_1) = \angle_q(p', \xi_1) = 3\pi/4$; the angle at v is even since the assumptions and Proposition 8.4 imply that $\eta_1\eta_2$ and $\eta_3\eta_4$ have the same type. It follows that the angle sum $\geq 7\pi/4$ or $\geq 11\pi/6$ depending on whether $R = (2, 4, 8)$ or $R = (2, 4, 6)$. Proposition 5.3 then implies $n(Q) \leq 2$. Since Q has an even angle at q, we conclude that $n(Q) = 2$ and (v, q, p') is the boundary of a chamber.

□

Proposition 8.13. *Suppose $R_1, R_2 \neq (2, 3, 8)$ are right triangles. Let $A \subset \Delta_1$ be an apartment and $\xi_1\xi_2, \eta_1\eta_2 \subset A$ two intersecting geodesics contained in the 1-skeleton of A. Then $\xi_1'\xi_2' \cap \eta_1'\eta_2' \neq \phi$.*

Proof. Denote $v = \xi_1\xi_2 \cap \eta_1\eta_2$. The proposition follows from Lemma 8.8 if $m(v) = 2$. From now on we assume $m(v) = 4, 6$ or 8. Suppose $\xi_1'\xi_2' \cap \eta_1'\eta_2' = \phi$. Proposition 8.4 implies $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ have different types. By Lemma 8.9, $\angle_v(\xi_1, \eta_1)$ is an odd angle. We may assume $\eta_1'\eta_2'$ contains vertices v' with $m(v') = 2$, and $\xi_1'\xi_2'$ does not. Let $\xi_3\xi_4, \eta_3\eta_4 \subset A^{(1)}$ be geodesics through v such that $\angle_v(\xi_1, \xi_3)$ and $\angle_v(\eta_1, \eta_3)$ are nonzero even angles. Lemma 8.9 implies $\xi_1'\xi_2'$ and $\xi_3'\xi_4'$ are of one type, while $\eta_1'\eta_2'$ and $\eta_3'\eta_4'$ are of another type. Since we have assumed $\xi_1'\xi_2' \cap \eta_1'\eta_2' = \phi$, Lemma 8.6 implies $\xi_3'\xi_4' \cap \eta_1'\eta_2' = \phi$, $\xi_3'\xi_4' \cap \eta_3'\eta_4' = \phi$ and $\xi_1'\xi_2' \cap \eta_3'\eta_4' = \phi$.

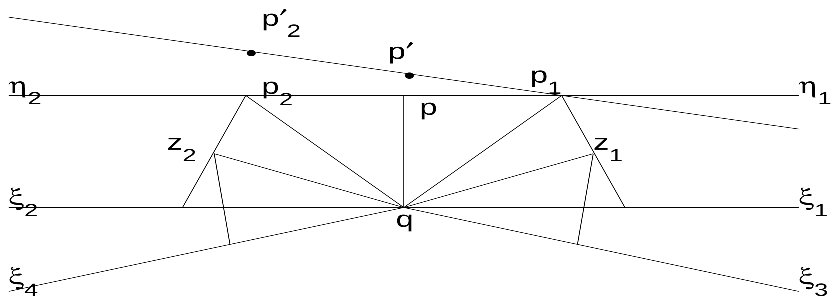

FIGURE 5.

Let pq ($p \in \eta_1'\eta_2'$, $q \in \xi_1'\xi_2'$) be the perpendicular between $\eta_1'\eta_2'$ and $\xi_1'\xi_2'$. Note $m(p) = 2, m(q) = 4$. Now Lemma 8.11 applied to $\eta_1'\eta_2'$ and $\xi_1'\xi_2'$, $\xi_3'\xi_4'$ implies $q \in \xi_3'\xi_4'$. Let $p_i \in p\eta_i'$ $(i = 1, 2)$ be the vertex adjacent to p. Denote by $\sigma, \tau_i, \sigma_j \in \mathrm{Link}(\Delta_2, q)$ $(i = 1, 2,\ j = 1, 2, 3, 4)$ the initial directions of qp, qp_i and $q\xi_j'$ respectively. Proposition 8.4 implies the map $f : \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \to \{\sigma\tau_1, \sigma\tau_2\}$, where $f(\sigma_j)$ is the initial edge of $\sigma\sigma_j \subset \mathrm{Link}(\Delta_2, q)$, is well-defined and satisfies the property: $f(\sigma_1) \neq f(\sigma_2)$, $f(\sigma_3) \neq f(\sigma_4)$. We may assume $f(\sigma_1) = f(\sigma_3) = \sigma\tau_1$ and $f(\sigma_2) = f(\sigma_4) = \sigma\tau_2$. Then $\sigma\tau_1 \subset \sigma\sigma_1 \cap \sigma\sigma_3$. The fact that $\xi_1'\xi_2'$ and $\xi_3'\xi_4'$ are at different sides implies that $0 < \angle_q(\xi_i', \xi_j') < \pi$ for $1 \leq i \leq 2$, $3 \leq j \leq 4$. Since $m(q) = 4$ and $\sigma\sigma_j$ has combinatorial length 3, we conclude that $\sigma\sigma_1 \cap \sigma\sigma_3$ has combinatorial length 2. Similarly $\sigma\sigma_2 \cap \sigma\sigma_4$ also has combinatorial length 2. Let z_1, z_2 be as shown in Figure 5, and $\omega_i \in \mathrm{Link}(\Delta_2, q)$ be the initial direction of qz_i. Then we have proved $\sigma\sigma_1 \cap \sigma\sigma_3 = \sigma\omega_1$, $\sigma\sigma_2 \cap \sigma\sigma_4 = \sigma\omega_2$. It follows that $\sigma_1\sigma_3 = \sigma_1\omega_1 \cup \omega_1\sigma_3$ and $\sigma_2\sigma_4 = \sigma_2\omega_2 \cup \omega_2\sigma_4$.

Now Lemma 8.12 applied to $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$, $\eta_3'\eta_4'$ implies that $p_i \in \eta_3'\eta_4'$ for $i = 1$ or 2. We may assume $p_1 \in \eta_3'\eta_4'$. Let $p'q'$ ($p' \in \eta_3'\eta_4'$, $q' \in \xi_1'\xi_2'$) be the perpendicular between $\eta_3'\eta_4'$ and $\xi_1'\xi_2'$. Then $q' = q$ and $T = (p_1, q, p')$ is the boundary of a chamber. Since $\eta_1'\eta_2'$, $\eta_3'\eta_4'$ are at different sides and $\angle_{p_1}(\eta_1', z_1) = \pi$, we have $p' \neq z_1$. Let $\sigma' \in \mathrm{Link}(\Delta_2, q)$ be the initial direction of qp'. Then $\sigma'\sigma_j = \sigma'\tau_1 \cup \tau_1\sigma_j$ for $j = 1, 3$. Let $p_2' \in \eta_3'\eta_4', p_2' \neq p_1$ be the vertex that is adjacent to p' but different from p_1, and $\tau_2' \in \mathrm{Link}(\Delta_2, q)$ the initial direction of qp_2'. Now the edge path $\sigma_2\sigma \cup \sigma\tau_1 \cup \tau_1\sigma' \cup \sigma'\sigma_2$ is a geodesic loop with length 2π in the generalized polygon $\mathrm{Link}(\Delta_2, q)$. It follows that this loop is injective. In particular, if ω_2' denotes the only vertex in the interior of $\sigma_2\tau_2'$, then $\omega_2' \neq \omega_2$. Now the argument in the second paragraph applied to $\eta_3'\eta_4'$ and $\xi_1'\xi_2'$, $\xi_3'\xi_4'$ implies that $\sigma_2\sigma_4 = \sigma_2\omega_2' \cup \omega_2'\sigma_4$. Consequently there are two different geodesic segments with length $\pi/2$ from σ_2 to σ_4: $\sigma_2\omega_2 \cup \omega_2\sigma_4$ and $\sigma_2\omega_2' \cup \omega_2'\sigma_4$. This contradicts to the fact that $\mathrm{Link}(\Delta_2, q)$ is a CAT(1) space.

□

For any $c \in \mathcal{D}_{A,v}$ we denote its image in $\mathcal{D}'_{A,v}$ by c'.

Lemma 8.14. *Proposition 8.5 holds for those v with $m(v) \neq 4$.*

Proof. Note there are $m(v)$ geodesics in $\mathcal{D}'_{A,v}$. By Proposition 8.13 any two geodesics in $\mathcal{D}'_{A,v}$ have nonempty intersection, in particular, Proposition 8.5 holds for those v with $m(v) = 2$. Suppose $m(v) = 6$ or 8. Lemma 8.9 implies that if $\xi_1\xi_2, \eta_1\eta_2 \in \mathcal{D}_{A,v}$ make even angles, then $\xi_1'\xi_2'$, $\eta_1'\eta_2'$ also make even angles. Fix three geodesics $c_1, c_2, c_3 \in \mathcal{D}_{A,v}$ that pairwise make nonzero even angles.

Since $R_2 \neq (2,3,8)$, Proposition 5.2 implies c'_1, c'_2, c'_3 intersect in a unique vertex $w \in \Delta_2$. Suppose there is a geodesic $c \in \mathcal{D}_{A,v}$ such that $w \notin c'$. Let $x_i = c'_i \cap c'$ $(i = 1,2,3)$. We may assume x_2 lies between x_1 and x_3. Consider the triangle (w, x_1, x_3): $x_2 \in \text{interior}(x_1x_3)$ and both $\angle_w(x_1, x_2)$ and $\angle_w(x_2, x_3)$ are even. Proposition 5.2 implies $R_2 = (2,3,8)$, contradicting to our assumption.

□

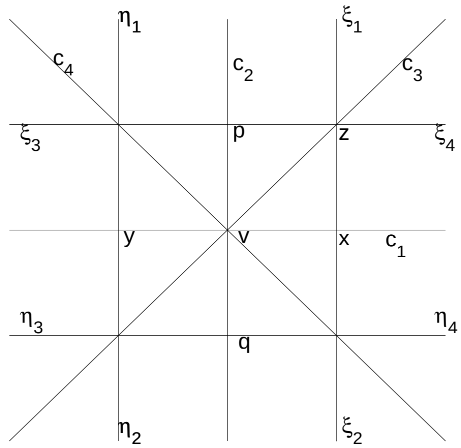

FIGURE 6.

Lemma 8.15. *Proposition 8.5 holds for those v with $m(v) = 4$.*

Proof. Let $c_1, c_2 \in \mathcal{D}_{A,v}$ be the two geodesics that contain vertices v' with $m(v') = 2$, and c_3, c_4 the other two geodesics in $\mathcal{D}_{A,v}$, see Figure 6. Let $x, y \in c_1$ be the two vertices on c_1 adjacent to v, and $p, q \in c_2$ the two vertices on c_2 adjacent to v, as shown in Figure 6. Let $\xi_1\xi_2, \eta_1\eta_2 \subset A^{(1)}$ be the two geodesics in A perpendicular to c_1 and passing through x, y respectively, and $\xi_3\xi_4, \eta_3\eta_4 \subset A^{(1)}$ the two geodesics in A perpendicular to c_2 and passing through p and q respectively. Let $w = c'_1 \cap c'_2$. Suppose $w \notin c'_3$. Denote $w_i = c'_3 \cap c'_i$ for $i = 1,2,4$, and $w'_2 = c'_2 \cap c'_4$. Since c_1, c_2 make even angles, Lemma 8.9 implies that c'_1, c'_2 make even angles at w. Proposition 5.2 applied to (w, w_1, w_2) shows that w_1w_2 contains exactly one vertex w_0 in the interior, and $m(w_0) = 2$ or 4. Now we observe $w_4 \in \{w_0, w_1, w_2\}$. Suppose otherwise, say, $w_1 \in \text{interior}(w_2w_4)$ holds. Then (w'_2, w_4, w_2) has an even angle at w_4 and one side w_2w_4 contains two vertices w_0 and w_1 in the interior, contradicting to Proposition 5.2.

We first consider the case $w_4 \in \{w_1, w_2\}$, say, $w_4 = w_1$. The triangle (w'_2, w_1, w_2) has an even angle at w_1. Since the side w_1w_2 contains w_0 in the interior, $m(w_0) = 2$ and $m(w_1) = 6$ or 8. Hence $\angle_{w_1}(w_2, w'_2) = \pi/3$ or $\pi/4$. The three geodesics c_1, c_3 and $\xi_1\xi_2$ have different types, and any two of them have nonempty intersection. Corollary 8.10 implies $w_1 \notin \xi'_1\xi'_2$. For the same reason, $w_1 \notin \eta'_1\eta'_2$. Let $z_i = c'_i \cap \xi'_1\xi'_2$ for $i = 3,4$. Then (w_1, z_3, z_4) has an even angle at w_1. Since $m(w_1) = 6$ or 8, $\angle_{w_1}(z_3, z_4) = \pi/3$ or $\pi/4$. It follows that either $\{w_2, w'_2\} = \{z_3, z_4\}$ or $\{w_2, w'_2\} \cap \{z_3, z_4\} = \phi$. If $\{w_2, w'_2\} = \{z_3, z_4\}$, then ξ'_1 is connected to an endpoint of c'_2 by a geodesic contained in $\Delta_2^{(1)}$. It follows that ξ_1 is connected to an endpoint of c_2 by a geodesic contained in $\Delta_1^{(1)}$, which is not true. Hence $\{w_2, w'_2\} \cap \{z_3, z_4\} = \phi$. Replacing $\xi'_1\xi'_2$ with $\eta'_1\eta'_2$ in the above argument, one sees $\eta'_1\eta'_2$ contains z_3 and z_4. Then η'_1 is connected to ξ'_1 or ξ'_2 by a geodesic in $\Delta_2^{(1)}$. It follows that η_1 is connected to ξ_1 or ξ_2 by a geodesic in $\Delta_1^{(1)}$, which is not true. The contradiction shows $w_4 \neq w_1$. Similarly $w_4 \neq w_2$. Hence $w_4 = w_0$. The triangle (w_0, w_2, w'_2) has an even angle at w_0. It follows that $m(w_0) \neq 2$ and hence $m(w_0) = 4$. Proposition 5.2 applied to (w_0, w_2, w'_2) implies

that $\angle_{w_2}(w_0, w_2') = \pi/6$ or $\pi/8$. Finally we apply Proposition 5.2 to (w, w_1, w_2) and conclude that $w = w_2'$.

Suppose $w_0 \notin \xi_1'\xi_2'$. Then the three geodesics c_3', c_4' and $\xi_1'\xi_2'$ form a triangle with an even angle at w_0. It follows that either $w_1 \in \xi_1'\xi_2'$ or $w_2 \in \xi_1'\xi_2'$. On the other hand, the preceding paragraph shows $w_1 \notin \xi_1'\xi_2'$. Hence $w_2 \in \xi_1'\xi_2'$. Let $w_5 = \xi_1'\xi_2' \cap c_4'$. Note $w_5 \neq w$, otherwise ξ_2' is connected to one of the endpoints of c_2' by a geodesic contained in $\Delta_2^{(1)}$, which is not true. Let $z = c_3 \cap \xi_1\xi_2$. Then $m(z) = 6$ or 8 and $z \in \xi_3\xi_4$. Since $w_2 = c_3' \cap \xi_1'\xi_2'$, Lemma 8.14 implies that $w_2 \in \xi_3'\xi_4'$. Let $w_6 = c_4' \cap \xi_3'\xi_4'$. Notice $w_6 \in ww_5$. Suppose otherwise, say $w_5 \in \text{interior}(ww_6)$ holds, then (w_2, w_5, w_6) has an angle $\angle_{w_5}(w_2, w_6) = 5\pi/6$ or $7\pi/8$, which is impossible. There are three vertices on ww_5: w, w_0, w_5. Assume $w_6 = w$. Then $\xi_3'\xi_4' \cap c_2'$ is a nontrivial interval, which implies ξ_3' is connected to an endpoint of c_2' by a geodesic contained in $\Delta_2^{(1)}$. It follows that ξ_3 is connected to an endpoint of c_2 by a geodesic contained in $\Delta_1^{(1)}$, which is not true. Similarly $w_6 \neq w_0, w_5$. The contradiction shows that $w_0 \in \xi_1'\xi_2'$. Similarly $w_0 \in \eta_1'\eta_2'$.

Since c_3 and $\xi_1\xi_2$ have different types, Lemma 8.9 implies c_3' and $\xi_1'\xi_2'$ have different types. Similarly c_3' and $\eta_1'\eta_2'$ have different types. So $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ have the same type. $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ make 4 angles at w_0: $\angle_{w_0}(\xi_1', \eta_1')$, $\angle_{w_0}(\xi_1', \eta_2')$, $\angle_{w_0}(\xi_2', \eta_1')$, $\angle_{w_0}(\xi_2', \eta_2')$. If any one of these angles is 0 or π, then some ξ_i' is connected to some η_j' by a geodesic contained in $\Delta_2^{(1)}$. It follows that ξ_i is connected to some η_j by a geodesic contained in $\Delta_1^{(1)}$, which is not true. Recall $m(w_0) = 4$. It follows that the 4 angles that $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ make at w_0 are all $\pi/2$. Hence $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ are locally contained in an apartment and Lemma 6.4 implies $\xi_1'\xi_2'$ and $\eta_1'\eta_2'$ are at different sides. Consequently, $\xi_1\xi_2$ and $\eta_1\eta_2$ are at different sides, which is not true. The contradiction shows $w \in c_3'$. Similarly $w \in c_4'$.

□

8.4. *Geodesics through a fixed vertex.*

Proposition 8.16. *Let Δ_1, Δ_2 be two Fuchsian buildings, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. Let $v \in \Delta_1$ be a vertex. If both R_1 and R_2 are right triangles different from $(2, 3, 8)$, then the geodesics in $\mathcal{D}_v'$ intersect in a unique vertex of Δ_2.*

Proof. Proposition 8.5 implies that for any vertex $v \in \Delta_1$ and any apartment A containing v, the geodesics in $\mathcal{D}_{A,v}'$ intersect in a unique vertex $v_A \in \Delta_2$. By Lemma 7.3, we only need to show that $v_{A_1} = v_{A_2}$ holds for any two apartments $A_1, A_2 \subset \Delta_1$ containing a vertex v with $\text{Link}(A_1, v) \cap \text{Link}(A_2, v)$ a half apartment in $\text{Link}(\Delta_1, v)$. Let $B_i = \text{Link}(A_i, v) (i = 1, 2)$, and ω_1, ω_2 the two endpoints of $B_1 \cap B_2$. Denote by m the midpoint of $B_1 \cap B_2$, and $m_i \in B_i$ the point in B_i opposite to m. Since $m(v)$ is even, m, m_1 and m_2 are all vertices in $\text{Link}(\Delta_1, v)$. Let $\xi_1, \xi_3 \in \partial A_1$ and $\xi_2, \xi_4 \in \partial A_2$ such that $v\xi_1$, $v\xi_2$ have initial direction ω_1 and $v\xi_3$, $v\xi_4$ have initial direction ω_2. Similarly let $\eta_1, \eta_3 \in \partial A_1$ and $\eta_2, \eta_4 \in \partial A_2$ such that $v\eta_3$, $v\eta_4$ have initial direction m, and $v\eta_i$ $(i = 1, 2)$ has initial direction m_i. Then we have $\eta_i\eta_j = v\eta_i \cup v\eta_j \subset \Delta_1^{(1)}$ if $i \neq j$, $\{i, j\} \neq \{3, 4\}$.

Now consider the three geodesics $\eta_1'\eta_2'$, $\eta_1'\eta_3'$, $\eta_3'\eta_2' \subset \Delta_2^{(1)}$. Since $R_2 \neq (2, 3, 8)$, Lemma 7.5 implies there is some vertex $w \in \Delta_2$ such that $\eta_1'\eta_2' = w\eta_1' \cup w\eta_2'$, $\eta_1'\eta_3' = w\eta_1' \cup w\eta_3'$ and $\eta_3'\eta_2' = w\eta_3' \cup w\eta_2'$. Notice that $\eta_i\eta_j$ $(1 \leq i \neq j \leq 3)$ and $\xi_1\xi_3$ make a right angle and are locally contained in an apartment. Lemma 6.4 implies $\eta_i\eta_j$ and $\xi_1\xi_3$ are at different sides. It follows that $\eta_i'\eta_j'$ and $\xi_1'\xi_3'$ are also at different sides. Now Lemma 8.7 implies $w \in \xi_1'\xi_3'$. Similarly $w \in \xi_2'\xi_4'$. Note $v_{A_1} = \xi_1'\xi_3' \cap \eta_1'\eta_3' = w$.

Now apply the above argument to η_1, η_2, η_4, instead of η_1, η_2, η_3, we see there is a vertex $w' \in \Delta_2$ such that $\eta_1'\eta_2' = w'\eta_1' \cup w'\eta_2'$, $\eta_1'\eta_4' = w'\eta_1' \cup w'\eta_4'$ and $\eta_4'\eta_2' = w'\eta_4' \cup w'\eta_2'$, and both $\xi_1'\xi_3'$ and $\xi_2'\xi_4'$ contain w'. In particular, $w' = \xi_1'\xi_3' \cap \eta_1'\eta_2' = w$. Therefore $v_{A_2} = \xi_2'\xi_4' \cap \eta_2'\eta_4' = w' = w = v_{A_1}$.

□

Proposition 8.17. *Let Δ_1, Δ_2 be two Fuchsian buildings, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. If one of R_1, R_2 is $(2,3,8)$, then so is the other.*

Proof. The proof is similar to that of Proposition 7.12. Suppose $R_1 = (2,3,8)$ and $R_2 \neq (2,3,8)$. Proposition 7.12 implies that R_2 is also a right triangle. Let A be an apartment of Δ_1. Let $\mathcal{G}$ be the set of geodesics contained in $A^{(1)}$ that do not contain any vertex v with $m(v) = 3$. Let $c_1, c_2 \in \mathcal{G}$ with $c_1 \cap c_2 \neq \phi$. Then c_1 and c_2 are at different sides and make even angles at $c_1 \cap c_2$. Lemma 8.9 implies that their images c_1', c_2' in Δ_2 have the same type. Note c_1', c_2' are also at different sides. Since $R_2 \neq (2,3,8)$, Proposition 8.4 implies $c_1' \cap c_2' \neq \phi$. It follows that c_1', c_2' make even angles at $c_1' \cap c_2'$.

The geodesics in $\mathcal{G}$ divide A into triangles, which shall be called "chambers". Let D be such a "chamber", and c_1, c_2, $c_3 \in \mathcal{G}$ the three geodesic containing the three sides of D. Since $R_2 \neq (2,3,8)$, Proposition 5.2 implies the images $c_i' \subset \Delta_2$ $(i = 1,2,3)$ have a common vertex, which we shall denote by w_D. Similarly, for a vertex $v \in A$ with $m(v) = 8$ and c_i $(i = 1,2,3,4)$ the 4 geodesics in $\mathcal{G}$ through v, Proposition 5.2 implies the images $c_i' \subset \Delta_2$ $(i = 1,2,3,4)$ have a common vertex, which is denoted by w_v.

Now let D_1, D_2 be two adjacent "chambers" in A. Then there is a vertex $v \in A$ and 5 geodesics c_i, $1 \leq i \leq 5$ such that c_1, c_2, c_3 pass through v, c_1, c_2, c_4 contain the sides of D_1, and c_2, c_3, c_5 contain the sides of D_2. The above paragraph shows $w_D = w_v = w_{D'}$. Notice A is gallery connected: given any two "chamber"s D, D', there is a finite sequence $D_0 = D, D_1, \cdots, D_k = D'$ such that D_i and D_{i+1} share a side. It follows that there is a vertex $w \in \Delta_2$ such that the images of the geodesics in $\mathcal{G}$ all contain w. We get a contradiction as in the proof of Proposition 7.12.

□

9. **The exceptional case** $(2,3,8)$**.** In this section we prove Theorem 1.3 when both R_1 and R_2 are $(2,3,8)$. As usual Proposition 5.2 is used in the manner as indicated in Remark 7.6.

9.1. *Buildings with chamber* $(2,3,8)$. Let Δ be a Fuchsian building with chamber $(2,3,8)$. The vertex links of Δ are generalized polygons. By Section 2.2 there are two integers $p, q \geq 2$, $p \neq q$ such that the following holds: an edge e is contained in exactly $p+1$ chambers if e is incident to a vertex v with $m(v) = 3$, and is contained in exactly $q+1$ chambers otherwise. There are only two types of complete geodesics in Δ:

(1) Type I: geodesics that do not contain any vertex v with $m(v) = 3$;
(2) Type II: geodesics that contain vertices v with $m(v) = 3$.

We shall use the following terminologies. We say an edge is *numbered* by i ($i = p$ or q) if it is contained in exactly $i+1$ chambers. All edges in Type I geodesics are numbered by q, and all edges in Type II geodesics are numbered by p. We say a vertex $v \in \Delta$ has *index* i ($i = 2, 3$ or 8) if $m(v) = i$; we also say v is *indexed* by i. If we record the indexes of vertices on a Type II geodesic in linear order, then they are: $\cdots, 8, 3, 2, 3, \cdots$, with period 4. We say the vertices on a Type II geodesic are *periodically indexed* by $8, 3, 2, 3$. Similarly if we record the indexes of vertices on a Type I geodesic in linear order, then they are: $\cdots, 8, 2 \cdots$, with period 2. We say the vertices on a Type I geodesic are periodically indexed by $8, 2$.

The following lemma follows directly from Proposition 5.2.

Lemma 9.1. *Let $T \subset \Delta^{(1)}$ be a triangle homeomorphic to a circle such that all the sides of T are contained in Type II geodesics. Then $S(T)$ is the union of two chambers as shown in Figure 3*(f).

Recall the chamber $(2,3,8)$ has area $A_0 = \pi/24$.

Lemma 9.2. *There is no quadrilateral $Q \subset \Delta^{(1)}$ with the following properties: all sides of Q are contained in Type II geodesics, and there is some side xy of Q such that the angles at x and y are $2\pi/3$.*

Proof. Suppose there is such a quadrilateral. Notice each angle of Q is $\geq \pi/4$. Proposition 5.3 implies that $n(Q) \leq 4$. But there are at least two chambers in $S(Q)$ incident to each of x and y. It follows that $S(Q)$ is the union of these 4 chambers. But the boundary of this union is not a quadrilateral, contradiction.

□

Lemma 9.3. *Let $Q = (x, y, z, w) \subset \Delta^{(1)}$ be a quadrilateral homeomorphic to a circle with the following properties: all sides of Q are contained in Type II geodesics, the angles at x and y are even and at least two vertices in the interior of xy are not indexed by 2. Then $m(v) \neq 8$ for every vertex $v \in$ interior(xy) and there is a vertex $v' \in zw$ with $m(v') = 8$.*

Proof. If $m(x) = 2$, then Q is actually a triangle and we obtain a contradiction to Lemma 9.1. Hence $m(x) \neq 2$, and similarly $m(y) \neq 2$. At least one of $m(x), m(y)$ is $\neq 3$, otherwise we have a contradiction to Lemma 9.2.

First assume exactly one of $m(x), m(y)$ is 3, say, $m(x) = 3$. Since the angle at x is even and each angle is at least $\pi/4$, Proposition 5.3 implies $n(Q) \leq 14$. Recall the vertices in a Type II geodesic are periodically indexed by $8, 3, 2, 3$. Since $m(x) = 3$, $m(y) = 8$ and at least two vertices in interior(xy) are not indexed by 2, there are vertices $v_1, v_2, v_3, v_4 \in$ interior(xy) such that x, v_1, v_2, v_3, v_4, y are in linear order and $m(v_1) = 8, m(v_2) = 3, m(v_3) = 2$, $m(v_4) = 3$. Recall the angles at x and y are even. We count the chambers in $S(Q)$: there are at least 2 chambers incident to x, 7 incident to v_1 but not x, 2 incident to v_2 but not v_1, 1 incident to v_3 but not v_2, 2 incident to v_4 but not v_3, and 1 incident to y but not v_4, for a total of 15. This contradicts to the above conclusion that $n(Q) \leq 14$.

Now assume $m(x) = m(y) = 8$. Suppose there is a vertex $v \in$ interior(xy) with $m(v) = 8$. Let $x_1 \in xy$ be the vertex on xy closest to x with $m(x_1) = 2$, and $y_1 \in xy$ the vertex on xy closest to y with $m(y_1) = 2$. Let C_1, C_2 be chambers in $S(Q)$ incident to x_1, y_1 respectively, and v_3, v_4 their vertices with $m(v_3) = m(v_4) = 8$ respectively. Note none of v_3, v_4 lies on the sides of Q incident to x or y because $\angle_x(v_3, y), \angle_y(v_4, x) = \pi/8$ and the angles of Q at x and y are even. It follows that $\{v_3, v_4\} \cap \{z, w\} = \phi$. Hence there are at least 8 chambers of $S(Q)$ incident to each of v_3, v_4, v. There are also at least two chambers incident to each of x and y. Therefore there are at least 28 chambers in $S(Q)$. On the other hand, since each angle of Q is at least $\pi/4$, Proposition 5.3 implies $n(Q) \leq 24$, a contradiction. Hence $m(v) \neq 8$ for all vertices $v \in$ interior(xy).

Since $m(v) \neq 8$ for all vertices $v \in$ interior(xy), there is exactly one vertex $v_0 \in xy$ with $m(v_0) = 2$. Let C be a chamber in $S(Q)$ incident to v_0, and v' the vertex of C with $m(v') = 8$. If $v' \in Q$, then the above argument shows $v' \in$ interior(zw) and the lemma holds. So we assume v' lies in the interior of $S(Q)$. Then there are at least 16 chambers in $S(Q)$ incident to v'. There are also (at least) two chambers incident to each of x, y. Hence there are at least 20 chambers in $S(Q)$. Suppose there is no vertex $v' \in zw$ with $m(v') = 8$. Then the angles at z and w are $\geq \pi/3$. Since the angles at x and y are $\geq \pi/4$, Proposition 5.3 implies $n(Q) \leq 20$. So $S(Q)$ must be the union of the 20 chambers exhibited above. However, the boundary of this union is not a quadrilateral, contradiction.

□

Lemma 9.4. *Let $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ $(\xi_1, \xi_2, \eta_1, \eta_2 \in \partial\Delta)$ be two geodesics that are locally contained in an apartment. If $\xi_1\xi_2$, $\eta_1\eta_2$ have the same type and make a right angle, then they are at different sides.*

Proof. Let $x_0 = \xi_1\xi_2 \cap \eta_1\eta_2$. The assumption implies $m(x_0) = 8$. If there is some $x \in \eta_1\eta_2$, $x \neq x_0$ such that $x\xi_1 \cap \xi_1\xi_2 = x'\xi_1$ $(x' \in \xi_1\xi_2)$ is a ray and $x\xi_1 \cap \eta_1\eta_2 = \{x\}$, then $(x', x_0, x) \subset \Delta^{(1)}$ has an even angle at x' and a right angle at x_0. Proposition 5.2 implies that $m(x_0) = 2$, contradicting to the above observation. Hence $x\xi_1 \cap \xi_1\xi_2 = \phi$ for any $x \in \eta_1\eta_2$, $x \neq x_0$. Similarly $x\xi_2 \cap \xi_1\xi_2 = \phi$ for any $x \in \eta_1\eta_2$, $x \neq x_0$ and $y\eta_i \cap \eta_1\eta_2 = \phi$ $(i = 1, 2)$ for any $y \in \xi_1\xi_2$, $y \neq x_0$. Then one argues as in Lemma 6.3 that $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides.

□

9.2. *Two geodesics at different sides.* Let Δ be a Fuchsian building with chamber $(2,3,8)$. There are two integers $p,q\geq 2$, $p\neq q$ such that all the edges in Type I geodesics are numbered by q, and all the edges in Type II geodesics are numbered by p. The goal of this section is to show the following:

Proposition 9.5. *Let Δ be a Fuchsian building with chamber $(2,3,8)$. If two disjoint geodesics $c_1,c_2\subset\Delta^{(1)}$ are at different sides, then they have different types.*

The proof of Proposition 9.5 is divided into two propositions (Propositions 9.6 and 9.12).

Proposition 9.6. *Let Δ be a Fuchsian building with chamber $(2,3,8)$. If two Type I geodesics $\xi_1\xi_2,\eta_1\eta_2\subset\Delta^{(1)}$ are at different sides, then $\xi_1\xi_2\cap\eta_1\eta_2\neq\phi$.*

Recall the vertices on a Type I geodesic are periodically indexed by $8,2$.

Lemma 9.7. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type I geodesics. If they are at different sides, then there are no vertices $y'\in\xi_1\xi_2$, $y\in\eta_1\eta_2$ such that $\angle_{y'}(y,\xi_1)=\angle_{y'}(y,\xi_2)=\pi$.*

Proof. Suppose there are vertices $y'\in\xi_1\xi_2$, $y\in\eta_1\eta_2$ such that $\angle_{y'}(y,\xi_1)=\angle_{y'}(y,\xi_2)=\pi$. The assumption and Lemma 6.6 implies that there exists $x_i\in\xi_1\xi_2$ $(i=1,2)$ such that $x_i\eta_i\cap\eta_1\eta_2=x_i'\eta_i$ $(x_i'\in\eta_1\eta_2)$ is a ray. We may assume $x_i\eta_i\cap\xi_1\xi_2=\{x_i\}$. We claim $\angle_y(y',\eta_i)=\pi/4$ for $i=1,2$. The triangle inequality then implies $\angle_y(\eta_1,\eta_2)\leq\pi/2$, contradicting to the fact that $y\in\eta_1\eta_2$. Next we prove the claim.

The fact that $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides implies that $\angle_y(y',\eta_i)<\pi$ for $i=1,2$. It follows that $x_i'\neq y$. The uniqueness of geodesics implies that $x_i'\in\text{interior}(y\eta_i)$. Notice $x_iy=x_iy'\cup y'y$ and (x_i,y,x_i') is a triangle. Since $\xi_1\xi_2$, $\eta_1\eta_2$ are Type I geodesics, all the angles of (x_i,y,x_i') are even. Proposition 5.2 then implies $\angle_y(y',\eta_i)=\angle_y(y',x_i')=\pi/4$.

□

Lemma 9.8. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type I geodesics. Suppose $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Let $y_i\in\eta_1\eta_2$ $(i=1,2)$ be a vertex such that $y_i\xi_i\cap\xi_1\xi_2=y_i'\xi_i$ is a ray and $y_i\xi_i\cap\eta_1\eta_2=\{y_i\}$. Then $y_1\neq y_2$.*

Proof. We suppose $y_1=y_2$. Lemma 9.7 implies $y_1'\neq y_2'$. The uniqueness of geodesic implies $y_1'\in\text{interior}(y_2'\xi_1)$. The triangle (y_1',y_2',y_1) has three even angles and Proposition 5.2 implies $\angle_{y_1}(y_1',y_2')=\pi/4$ and y_1y_j' is the union of two edges. Let $x_i\in\xi_1\xi_2$ be such that $x_i\eta_i\cap\eta_1\eta_2=x_i'\eta_i$ $(i=1,2)$ is a ray and $x_i\eta_i\cap\xi_1\xi_2=\{x_i\}$.

First suppose $x_i\notin y_1'y_2'$ for $i=1$ or 2. Then $x_i\in\text{interior}(y_j'\xi_j)$ for some j. The uniqueness of geodesic implies $x_i'\in\text{interior}(y_1\eta_i)$. Then (y_1,x_i,x_i') has three even angles and has a side $y_1x_i=y_1y_j'\cup y_j'x_i$ consisting of at least three edges, contradicting to Proposition 5.2. Hence $x_i\in y_1'y_2'$ for $i=1,2$. Note $x_i\eta_i$ is of Type I since it shares a ray with a Type I geodesic. It follows that x_ix_i' makes even angles with $\xi_1\xi_2$. Hence $m(x_i)\neq 2$, otherwise $\angle_{x_i}(\xi_1,x_i')=\angle_{x_i}(\xi_2,x_i')=\pi$, contradicting to Lemma 9.7. On the other hand, $y_1'y_2'$ contains only one vertex v in the interior and it satisfies $m(v)=2$. Hence $\{x_1,x_2\}\subset\{y_1',y_2'\}$. In particular, there is some j with $x_1=y_j'$. The uniqueness of geodesic implies $x_1'\in y_1\eta_1$. The equality $x_1'=y_1$ does not hold since it would imply ξ_j and η_1 are connected by a geodesic contained in the 1-skeleton, contradicting to the assumption that $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Now (x_1,y_1,x_1') has three even angles and so y_1x_1' is the union of two edges and $\angle_{y_1}(y_j',\eta_1)=\pi/4$. Similarly $x_2=y_l'$ for some $l=1,2$, $x_2'\in\text{interior}(y_1\eta_2)$ and $\angle_{y_1}(y_l',\eta_2)=\pi/4$. If $l=j$, then triangle inequality implies $\angle_{y_1}(\eta_1,\eta_2)\leq\pi/2$, contradiction to the fact that $y_1\in\eta_1\eta_2$. If $l\neq j$, then $\angle_{y_1}(\eta_1,\eta_2)\leq\angle_{y_1}(y_j',\eta_1)+\angle_{y_1}(y_1',y_2')+\angle_{y_1}(y_l',\eta_2)\leq 3\pi/4$, again a contradiction.

□

The index $i+1$ is taken mod 2 in the proofs of Lemma 9.9, Propositions 9.6 and 9.12.

Lemma 9.9. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type I geodesics. Suppose $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Let $y_i \in \eta_1\eta_2$ $(i = 1, 2)$ be such that $y_i\xi_i \cap \xi_1\xi_2 = y_i'\xi_i$ is a ray. Then $y_1' \neq y_2'$.*

Proof. We suppose $y_1' = y_2'$. We may assume $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. The triangle (y_1', y_1, y_2) has three even angles, and hence each side is the union of two edges and $\angle_{y_i}(y_1', y_{i+1}) = \pi/4$. Let $x_1 \in \xi_1\xi_2$ be such that $x_1\eta_1 \cap \eta_1\eta_2 = x_1'\eta_1$ is a ray and $x_1\xi_1 \cap \xi_1\xi_2 = \{x_1\}$. First assume $x_1 \in \text{interior}(y_1'\xi_1)$. The uniqueness of geodesic shows $x_1' \in \text{interior}(y_1\eta_1)$. Then (x_1', y_1, x_1) has three even angles and its side $x_1y_1 = x_1y_1' \cup y_1'y_1$ contains at least three edges, contradicting to Proposition 5.2. Similarly $x_1 \in \text{interior}(y_1'\xi_2)$ cannot hold, and we must have $x_1 = y_1'$. Uniqueness of geodesic implies $x_1' \notin \text{interior}(y_i\eta_2)$ for $i = 1, 2$. The fact that $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides implies $x_1' \neq y_1', y_2'$. There is some $i = 1, 2$ such that $x_1' \in \text{interior}(y_i\eta_1)$ and $y_{i+1} \notin y_i\eta_1$. Then (x_1, x_1', y_i) has three even angles and hence we have $\angle_{y_i}(x_1', x_1) = \pi/4$. It follows that $\angle_{y_i}(\eta_1, \eta_2) \leq \pi/2$, contradicting to the fact that $y_i \in \eta_1\eta_2$.

□

Lemmas 9.8, 9.9 and uniqueness of geodesic imply the following:

Corollary 9.10. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type I geodesics. Suppose $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Let $y_i \in \eta_1\eta_2$ $(i = 1, 2)$ be such that $y_i\xi_i \cap \xi_1\xi_2 = y_i'\xi_i$ is a ray and $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. Then $y_1 \neq y_2$ and $y_1' \in$ interior$(y_2'\xi_1)$.*

Lemma 9.11. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type I geodesics. Suppose $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Let $y_1 \in \eta_1\eta_2$, $x_1 \in \xi_1\xi_2$ be such that $y_1\xi_1 \cap \xi_1\xi_2 = y_1'\xi_1$, $x_1\eta_1 \cap \eta_1\eta_2 = x_1'\eta_1$ are rays and $y_1\xi_1 \cap \eta_1\eta_2 = \{y_1\}$, $x_1\eta_1 \cap \xi_1\xi_2 = \{x_1\}$. Then $y_1' \in$ interior$(x_1\xi_1)$ and $x_1' \in$ interior$(y_1\eta_1)$.*

Proof. We show (1) and (2) of Proposition 6.8 cannot occur. First suppose (1) of Proposition 6.8 occurs, that is, $y_1' \in \text{interior}(x_1\xi_1)$ and $y_1 \in \text{interior}(x_1'\eta_1)$. In this case, $x_1y_1 = x_1x_1' \cup x_1'y_1$ and (y_1', y_1, x_1) has three even angles. It follows that the only vertex x_1' in the interior of x_1y_1 satisfies $m(x_1') = 2$. Since $\angle_{x_1'}(\eta_1, x_1) = \pi$, we also have $\angle_{x_1'}(\eta_2, x_1) = \pi$, contradicting to Lemma 9.7.

Now suppose (2) of Proposition 6.8 occurs, that is, $y_1' \in \text{interior}(x_1\xi_1)$ and $y_1 = x_1'$. Let $x_2 \in \xi_1\xi_2$, $y_2 \in \eta_1\eta_2$ be such that $x_2\eta_2 \cap \eta_1\eta_2 = x_2'\eta_2$, $y_2\xi_2 \cap \xi_1\xi_2 = y_2'\xi_2$ are rays and $x_2\eta_2 \cap \xi_1\xi_2 = \{x_2\}$, $y_2\xi_2 \cap \eta_1\eta_2 = \{y_2\}$. Corollary 9.10 implies $x_2' \in \text{interior}(x_1'\eta_2)$ and $y_2' \in \text{interior}(y_1'\xi_2)$. Notice $m(x_i) = m(x_i') = m(y_i) = m(y_i') = 8$ for $i = 1, 2$, otherwise if, say, $m(x_1) = 2$, then $\angle_{x_1}(x_1', \xi_i) = \pi$ for $i = 1, 2$, contradicting to Lemma 9.7. Since (x_1, x_1', y_1') has three even angles, x_1y_1' contains no vertex indexed by 8 in the interior. So we have $y_2' \in x_1\xi_2$.

First consider the case $y_2' = x_1$. Then $y_2 \neq y_1$ by Lemma 9.8. Since $\angle_{x_1'}(x_1, \eta_1) = \pi$, we have $y_2 \in \text{interior}(y_1\eta_2)$. The triangle (x_1, y_1, y_2) has three even angles and so the side y_1y_2 contains no vertex indexed by 8 in the interior. Consider x_2x_2'. Corollary 9.10 implies $x_2 \neq x_1$ and $x_2' \in \text{interior}(x_1'\eta_2)$. Since $m(x_2') = 8$, we have $x_2' \in y_2\eta_2$. By the first paragraph $x_2 \in \text{interior}(x_1\xi_1)$. Since $m(x_2) = 8$ and x_1y_1' contains no vertex indexed by 8 in the interior, $x_2 \in y_1'\xi_1$. Now $x_2y_1 = x_2y_1' \cup y_1'y_1$ and (x_2, y_1, x_2') has three even angles. The three triangles (x_2, y_1, x_2'), (x_1', y_1', x_1) and (x_1, y_2, y_1) give rise to a loop with length $3\pi/4$ in $\text{Link}(\Delta, y_1)$, which is impossible.

Now consider the case $y_2' \in \text{interior}(x_1\xi_2)$. By considering (x_1, y_1, y_1') we see $\angle_{x_1}(y_1, y_1') = \pi/4$ and so $\angle_{x_1}(y_1, \xi_2) \geq 3\pi/4$. It follows that $y_2 \in \text{interior}(y_1\eta_2)$. Consider $Q = (x_1, y_1, y_2, y_2')$. Q has one angle $\angle_{x_1}(y_1, \xi_2) \geq 3\pi/4$ and the other three angles $\geq \pi/4$. Proposition 5.3 implies $n(Q) \leq 12$. On the other hand, there are at least 6 chambers in $S(Q)$ incident to x_1, at least 2 chambers incident to each of y_2', y_2, x_1', for a total of 12. It follows that $S(Q)$ is the union of these 12 chambers. However, the boundary of this union is not a quadrilateral, contradiction.

□

Proof of Proposition 9.6. Suppose $\xi_1\xi_2 \cap \eta_1\eta_2 = \phi$. Let $y_i \in \eta_1\eta_2$ and $x_i \in \xi_1\xi_2$ $(i = 1, 2)$ be such that $y_i\xi_i \cap \xi_1\xi_2 = y_i'\xi_i$, $x_i\eta_i \cap \eta_1\eta_2 = x_i'\eta_i$ are rays and $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$, $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$. Corollary 9.10 and Lemma 9.11 imply that there are two cases:

(1) there is some $i = 1, 2$ such that $\xi_1, y_1', x_i, x_{i+1}, y_2', \xi_2$ and $\eta_i, x_i', y_1, y_2, x_{i+1}', \eta_{i+1}$ are in linear order;
(2) there is some $i = 1, 2$ such that $\xi_1, y_1', x_i, x_{i+1}, y_2', \xi_2$ and $\eta_i, x_i', y_2, y_1, x_{i+1}', \eta_{i+1}$ are in linear order.

We prove (2) is impossible, one similarly proves that (1) is impossible. Suppose (2) holds. Recall $m(v) = 8$ if v is one of the vertices x_i, x_i', y_i, y_i' $(i = 1, 2)$. Consider $Q = (y_1', y_1, y_2, y_2')$. Since all the angles of Q are even, Proposition 5.3 implies $n(Q) \leq 24$. We count the chambers in $S(Q)$ by counting the chambers incident to vertices indexed by 8. The vertices x_i, x_{i+1} lie in the interior of $y_1'y_2'$, and so there are 8 chambers incident to each of them. There are also at least two chambers incident to each of y_1, y_1', y_2, y_2'. We have exhibited 24 chambers in $S(Q)$. It follows that $n(Q) = 24$ and $\angle_{y_1}(y_1', y_2) = \pi/4$. Now consider $Q' = (y_1', y_1, x_{i+1}', x_{i+1})$. The quadrilateral Q' has an angle $\angle_{y_1}(y_1', x_{i+1}') \geq 3\pi/4$. It follows that $m(Q') \leq 12$. On the other hand, there are at least 8 chambers in $S(Q')$ incident to x_i and at least 2 chambers incident to each of $y_1', y_1, x_{i+1}', x_{i+1}$, for a total of 16, contradiction.

□

We next consider Type II geodesics.

Proposition 9.12. *Let Δ be a Fuchsian building with chamber $(2, 3, 8)$. If two Type II geodesics $\xi_1\xi_2, \eta_1\eta_2 \subset \Delta^{(1)}$ are at different sides, then $\xi_1\xi_2 \cap \eta_1\eta_2 \neq \phi$.*

We start with some lemmas.

Lemma 9.13. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type II geodesics. Suppose $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Let $y_i \in \eta_1\eta_2$ $(i = 1, 2)$ be such that $y_i\xi_i \cap \xi_1\xi_2 = y_i'\xi_i$ is a ray and $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. Then $y_1 \neq y_2$.*

Proof. Suppose $y_1 = y_2$. If $y_1' \neq y_2'$, then $y_1' \in \text{interior}(y_2'\xi_1)$. (y_1', y_2', y_1) has two even angles and all its sides are contained in Type II geodesics, contradicting to Lemma 9.1. So we have $y_1' = y_2'$. In this case, $\angle_{y_1'}(\xi_1, y_1) = \angle_{y_1'}(\xi_2, y_1) = \pi$. Let $x_i \in \xi_1\xi_2$ $(i = 1, 2)$ be such that $x_i\eta_i \cap \eta_1\eta_2 = x_i'\eta_i$ is a ray and $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$. The uniqueness of geodesic implies $x_i' \in y_1\eta_i$. Since $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides, we have $x_i' \neq y_1$. Lemma 9.1 applied to (x_i, x_i', y_1) shows $\angle_{y_1}(\eta_i, y_1') = \angle_{y_1}(x_i', y_1') = \pi/4$ or $\pi/3$. Triangle inequality then implies $\angle_{y_1}(\eta_1, \eta_2) \leq 2\pi/3$, contradicting to the fact that $y_1 \in \eta_1\eta_2$.

□

Lemma 9.14. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type II geodesics. Suppose $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Let $y_1 \in \eta_1\eta_2$, $x_1 \in \xi_1\xi_2$ be such that $y_1\xi_1 \cap \xi_1\xi_2 = y_1'\xi_1$, $x_1\eta_1 \cap \eta_1\eta_2 = x_1'\eta_1$ are rays and $y_1\xi_1 \cap \eta_1\eta_2 = \{y_1\}$, $x_1\eta_1 \cap \xi_1\xi_2 = \{x_1\}$. Then $y_1' \in$ interior$(x_1\xi_1)$ and $x_1' \in$ interior$(y_1\eta_1)$.*

Proof. We show (1) and (2) of Proposition 6.8 cannot occur. First suppose (1) of Proposition 6.8 occurs, that is, $y_1' \in \text{interior}(x_1\xi_1)$ and $y_1 \in \text{interior}(x_1'\eta_1)$. In this case, all sides of (x_1, y_1, y_1') are contained in Type II geodesics and one of them $x_1y_1 = x_1x_1' \cup x_1'y_1$ consists of (at least) two edges. Lemma 9.1 implies $m(x_1') = 2$. It follows that $\angle_{x_1'}(x_1, \eta_1) = \angle_{x_1'}(x_1, \eta_2) = \pi$, contradicting to Lemma 9.13.

Now suppose (2) of Proposition 6.8 occurs, that is, $y_1' \in \text{interior}(x_1\xi_1)$ and $y_1 = x_1'$. The triangle (x_1, y_1, y_1') has an even angle at y_1'. Lemma 9.1 implies $y_1'y_1$, $y_1'x_1$ are edges in Δ, $m(y_1') = 8$, $m(y_1) = 3$ and $\angle_{x_1}(y_1, y_1') = \pi/3$. Let $x_2 \in \xi_1\xi_2$ be such that $x_2\eta_2 \cap \eta_1\eta_2 = x_2'\eta_2$ is a ray and $x_2\eta_2 \cap \xi_1\xi_2 = \{x_2\}$. Lemma 9.13 implies $x_1 \neq x_2$. The uniqueness of geodesic implies $x_2' \in x_1'\eta_2$. First assume $x_2' = x_1'$. Since $\angle_{y_1'}(\xi_1, y_1) = \pi$, the uniqueness of geodesic implies $x_2 \in y_1'\xi_2$. The fact that $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides implies $x_2 \neq y_1'$. Hence $x_2 \in \text{interior}(x_1\xi_2)$. In this case, (x_1, x_2, y_1) has an angle $\angle_{x_1}(x_2, y_1) \geq 2\pi/3$, contradicting to Lemma 9.1. Therefore we must have $x_2' \in \text{interior}(x_1'\eta_2)$. If $x_2 \in \text{interior}(y_1'\xi_1)$, then the side x_2y_1 of (x_2, y_1, x_2') contains y_1' in the interior with $m(y_1') = 8$, contradicting to Lemma 9.1. If $x_2 = y_1'$, then (x_2, y_1, x_2') has an even angle at x_2' and Lemma 9.1 implies $y_1'y_1$ consists of two edges, contradicting to the above observation that $y_1'y_1$

is an edge. The only remaining possibility is $x_2 \in \text{interior}(x_1\xi_2)$. In this case, the two angles of (x_2, x_1, y_1, x_2') at x_1 and y_1 are $2\pi/3$, contradicting to Lemma 9.2.

□

Lemma 9.15. *Let $\xi_1\xi_2$, $\eta_1\eta_2$ be two disjoint Type II geodesics. Suppose $\xi_1\xi_2$, $\eta_1\eta_2$ are at different sides. Let $y_i \in \eta_1\eta_2$ $(i = 1, 2)$ be such that $y_i\xi_i \cap \xi_1\xi_2 = y_i'\xi_i$ is a ray and $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$. Then $y_1 \neq y_2$ and $y_1' \in$ interior$(y_2'\xi_1)$.*

Proof. The claim $y_1 \neq y_2$ is the content of Lemma 9.13. Let $x_1 \in \xi_1\xi_2$ be such that $x_1\eta_1 \cap \eta_1\eta_2 = x_1'\eta_1$ is a ray and $x_1\eta_1 \cap \xi_1\xi_2 = \{x_1\}$. Lemma 9.14 implies $y_1' \in \text{interior}(x_1\xi_1)$. The same lemma applied to $y_2\xi_2$ and $x_1\eta_1$ shows $y_2' \in \text{interior}(x_1\xi_2)$. Hence $y_1' \in \text{interior}(y_2'\xi_1)$.

□

Proof of Proposition 9.12. Suppose $\xi_1\xi_2 \cap \eta_1\eta_2 = \phi$. Let $y_i \in \eta_1\eta_2$ and $x_i \in \xi_1\xi_2$ $(i = 1, 2)$ be such that $y_i\xi_i \cap \xi_1\xi_2 = y_i'\xi_i$, $x_i\eta_i \cap \eta_1\eta_2 = x_i'\eta_i$ are rays and $y_i\xi_i \cap \eta_1\eta_2 = \{y_i\}$, $x_i\eta_i \cap \xi_1\xi_2 = \{x_i\}$. Lemma 9.14 and Lemma 9.15 imply that there are two possibilities:
(1) there is some $i = 1, 2$ such that $\xi_1, y_1', x_i, x_{i+1}, y_2', \xi_2$ and $\eta_i, x_i', y_1, y_2, x_{i+1}', \eta_{i+1}$ are in linear order;
(2) there is some $i = 1, 2$ such that $\xi_1, y_1', x_i, x_{i+1}, y_2', \xi_2$ and $\eta_i, x_i', y_2, y_1, x_{i+1}', \eta_{i+1}$ are in linear order.
We prove (1) is impossible, the proof that (2) is impossible is similar.

We assume (1) holds. We first observe that $m(v) \neq 2$ if v is one of the x_i, y_i, x_i', y_i': if, for example, $m(x_1') = 2$, then $\angle_{x_1'}(\eta_1, x_1) = \angle_{x_1'}(\eta_2, x_1) = \pi$, contradicting to Lemma 9.13. Consider the quadrilateral $(x_i, x_{i+1}, x_{i+1}', x_i')$. The two angles at x_i' and x_{i+1}' are even, and the side $x_i'x_{i+1}'$ contains the two vertices y_1, y_2 in the interior with $m(y_1), m(y_2) \neq 2$. Lemma 9.3 implies $m(v) \neq 8$ for every vertex $v \in \text{interior}(x_i'x_{i+1}')$ and there is a vertex $v' \in x_ix_{i+1}$ with $m(v') = 8$. But the same lemma applied to (y_1', y_2', y_2, y_1) implies $m(v) \neq 8$ for every vertex $v \in \text{interior}(y_1'y_2')$. Here we have a contradiction since $x_ix_{i+1} \subset \text{interior}(y_1'y_2')$ and $v' \in x_ix_{i+1}$ with $m(v') = 8$.

□

9.3. *The image of an apartment.* Let Δ_1 and Δ_2 be two Fuchsian buildings with chamber $(2, 3, 8)$, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. There are integers $p \neq q \geq 2$ such that all edges in Δ_1 incident to vertices v with $m(v) = 3$ are contained in exactly $p+1$ chambers, and all other edges are contained in exactly $q+1$ chambers. By Lemma 7.1 there are two possibilities:
(1) A geodesic $c \subset \Delta_1^{(1)}$ is Type I if and only if its image c' is Type I. Hence all edges in Δ_2 incident to vertices w with $m(w) = 3$ are contained in exactly $p + 1$ chambers, and all other edges are contained in exactly $q + 1$ chambers;
(2) A geodesic $c \subset \Delta_1^{(1)}$ is Type I if and only if its image c' is Type II. Hence all edges in Δ_2 incident to vertices w with $m(w) = 3$ are contained in exactly $q + 1$ chambers, and all other edges are contained in exactly $p + 1$ chambers.

Lemma 9.16. *Let A be an apartment in Δ_1, and $v \in A$ a vertex with $m(v) = 8$. Then the images of the 4 Type I geodesics (in A) through v intersect in a unique vertex of Δ_2.*

Proof. Let $\gamma_1, \gamma_2, \gamma_3 \subset \Delta_2^{(1)}$ be three geodesics of the same type. Suppose any two of them intersect in a vertex and $\gamma_1 \cap \gamma_2 \cap \gamma_3 = \phi$. Denote $x_i = \gamma_{i-1} \cap \gamma_{i+1}$ $(i \mod 3)$ and apply Proposition 5.2 to $T = (x_1, x_2, x_3)$. If the γ_i's are Type I, then all the angles of T are $\pi/4$; if the γ_i's are Type II, then the angles of T are $\pi/3$, $\pi/3$ and $\pi/4$. In any case, the angles of T are $< \pi/2$.

Now let $c_i \subset A$ $(i = 1, 2, 3, 4)$ be the 4 Type I geodesics through v, and c_i' their images. Then the geodesics c_i', $1 \leq i \leq 4$ have the same type. Proposition 9.5 implies $c_i' \cap c_j' \neq \phi$ for $1 \leq i, j \leq 4$. First suppose c_1', c_2', c_3' intersect in a point w and $w \notin c_4'$. Let $y_i = c_4' \cap c_i'$ for $1 \leq i \leq 3$. We may

assume y_2 lies between y_1 and y_3. Consider (w, y_1, y_2) and (w, y_2, y_3). The first paragraph implies $\angle_{y_2}(w, y_1) < \pi/2$ and $\angle_{y_2}(w, y_3) < \pi/2$. The triangle inequality implies that $\angle_{y_2}(y_1, y_3) < \pi$, contradicting to $y_2 \in \text{interior}(y_1y_3)$.

Next we assume any three of c_i', $1 \leq i \leq 4$ have empty intersection. Let y_i, $i = 1, 2, 3$ be as above. We may assume y_2 lies between y_1 and y_3. Let $w = c_1' \cap c_3'$ and $w_1 = c_1' \cap c_2'$. Consider (w_1, y_1, y_2). The first paragraph implies $\angle_{y_2}(w_1, y_1) < \pi/2$. In particular, $m(y_2) \neq 2$. Since $y_2 \in \text{interior}(y_1y_3)$, Proposition 5.2 applied to (w, y_1, y_3) implies $m(y_2) = 2$, a contradiction.

$\square$

Let A be an apartment in Δ_1. Type I geodesics in A divide A into triangular regions, each of which is the union of 6 chambers. We consider the triangulation of A whose 1-skeleton is the union of Type I geodesics. We denote by $A^{[i]}$, $i = 0, 1, 2$ the i-skeleton of this triangulation. Notice $A^{[0]}$ is the set of vertices v in A with $m(v) = 8$.

Proposition 9.17. *Let Δ_1 and Δ_2 be two Fuchsian buildings with chamber $(2, 3, 8)$, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. Then a geodesic $c \subset \Delta_1^{(1)}$ is Type I if and only if its image c' is Type I.*

Proof. We suppose the opposite holds, that is, a geodesic $c \subset \Delta_1^{(1)}$ is Type I if and only if its image c' is Type II. Fix an apartment A of Δ_1. First suppose there is a 2-simplex D in $A^{[2]}$ such that $c_1' \cap c_2' \cap c_3' \neq \phi$, where $c_1, c_2, c_3 \subset A$ are the three geodesics that contain the edges of D. Then Lemma 9.16 and the proof of Proposition 8.17 imply that the images of all the Type I geodesics in A intersect in a unique vertex of Δ_2, which is impossible. Hence for any 2-simplex D in $A^{[2]}$, if $c_1, c_2, c_3 \subset A$ are the three geodesics that contain the edges of D, then $c_1' \cap c_2' \cap c_3' = \phi$. Since c_1', c_2', c_3' are Type II geodesics, Lemma 9.1 implies they form a triangle in Δ_2 which has two angles $= \pi/3$ and one angle $= \pi/4$.

Let D_1, D_2 be two 2-simplices in $A^{[2]}$ that share an edge. Let v be one of the common vertices of D_1 and D_2, c_2 the geodesic in A containing the common edge of D_1 and D_2, $c_1 \subset A^{(1)}$ the geodesic containing the other edge of D_1 incident to v, $c_3 \subset A^{(1)}$ the geodesic containing the other edge of D_2 incident to v, and γ_i $(i = 1, 2)$ the geodesic containing the third edge of D_i. Denote $w = c_1' \cap c_2'$, $x_i = c_i' \cap \gamma_i'$ $(i = 1, 2)$, $x_3 = c_3' \cap \gamma_2'$. By the last paragraph, $T_i = (w, x_i, x_{i+1})$ is a triangle with two vertices indexed by 3. We may assume $m(w) = 3$. Notice $S(T_1) \cap S(T_2) \subset \Delta_2$ is a convex subcomplex containing wx_2. We claim $S(T_1) \cap S(T_2) = wx_2$. By Lemma 9.1 $S(T_i)$ is the union of two chambers. If $S(T_1) \cap S(T_2) \neq wx_2$, then $S(T_1) \cap S(T_2)$ contains a chamber incident to w, which implies $c_1' \cap c_3'$ contains an edge of Δ_2. This is impossible since c_1' and c_3' are at different sides, see Lemma 6.12.

Let $c_4 \subset A$ be the remaining Type I geodesic through v. There is a 2-simplex D_i $(i = 3, 4)$ of $A^{[2]}$ incident to v and with edges lying on c_i and c_{i+1} ($i \mod 4$). Let γ_i be the geodesic containing the third edge of D_i, $i = 3, 4$. Denote $x_4 = c_4' \cap \gamma_4'$ and $x_5 = c_1' \cap \gamma_4'$. There are 4 triangles $T_i = (w, x_i, x_{i+1})$, $1 \leq i \leq 4$. Notice $x_1, x_5 \in c_1'$ and $d(w, x_1) = d(w, x_5)$. Hence there are two possibilities: either $x_5 = x_1$ or w is the midpoint of x_1x_5. If $x_5 = x_1$, then the claim in the last paragraph implies that the 4 triangles T_i give rise to a geodesic loop with length $\frac{\pi}{3} \times 4$ in the CAT(1) space $\text{Link}(\Delta_2, w)$, a contradiction. If w is the midpoint of x_1x_5, then the same claim implies that the T_i's give rise to an edge path with length 4 between two opposite vertices of the generalized polygon $\text{Link}(\Delta_2, w)$. This is impossible because $\text{Link}(\Delta_2, w)$ is a generalized 3-gon and opposite vertices have different types.

$\square$

Lemma 9.16 implies for any apartment $A \subset \Delta_1$, there is a well-defined map $f_A : A^{[0]} \to \Delta^{(0)}$, where for any $v \in A^{[0]}$, $f_A(v)$ is the unique intersection point of the images of the 4 Type I geodesics in A through v. We denote $f_A(v)$ by v'.

Proposition 9.18. *Let A be an apartment in Δ_1. Then there is a unique apartment A' in Δ_2 with the following properties:*
(1) $f_A(A^{[0]}) = {A'}^{[0]}$;
(2) *the map* $f_A : A^{[0]} \to {A'}^{[0]}$ *extends to an isomorphism* $g_A : A \to A'$.

Proof. Let D be a 2-simplex in $A^{[2]}$. Denote by $c_1, c_2, c_3 \subset A^{(1)}$ the three geodesics that contain the three edges of D. If $c_1' \cap c_2' \cap c_3' \neq \phi$, then Lemma 9.16 and the proof of Proposition 8.17 imply that the images of all the Type I geodesics in A have nonempty intersection, impossible. Hence, $c_1' \cap c_2' \cap c_3' = \phi$. Denote $x_i = c_{i-1} \cap c_{i+1}$ ($i \mod 3$). Since c_i' is a Type I geodesic, $T' = (x_1', x_2', x_3')$ has three even angles and Proposition 5.2 implies that $S(T')$ is isometric to D. We define g_D to be the unique isometry extending the map $f_A|_{\{x_1,x_2,x_3\}}$. It is clear that if D_1, $D_2 \subset A^{[2]}$ are two 2-simplices with $D_1 \cap D_2 \neq \phi$, then g_{D_1} and g_{D_2} agree on $D_1 \cap D_2$. Hence we have a map $g_A : A \to \Delta_2$, where $g_A|_D = g_D$ for each 2-simplex $D \subset A^{[2]}$.

Consider two arbitrary 2-simplices D_1, $D_2 \subset A^{[2]}$ that share an edge. Let vx_2 ($v, x_2 \in A^{[0]}$) be the common edge of D_1 and D_2, x_1 the third vertex of D_1 and x_3 the third vertex of D_2. Let $c_i \subset A$ ($i = 1, 2, 3$) be the geodesic containing vx_i, and $\gamma_j \subset A$ ($j = 1, 2$) the geodesic containing $x_j x_{j+1}$. Let $T_i' = (w, x_i', x_{i+1}')$ ($i = 1, 2$). Notice image$(g_{D_i}) = S(T_i')$. We claim image$(g_{D_1}) \cap$image$(g_{D_2}) = v'x_2'$. The claim implies that g_A is an isometry into Δ_2 and hence image(g_A) is an apartment in Δ_2. Next we prove the claim.

Suppose the claim is false. Recall $S(T_i') \subset \Delta_2$ is a convex subcomplex and is the union of 6 chambers. We observe that $S(T_1') \cap S(T_2')$ is the union of the two chambers in $S(T_1')$ that have edges lying on $v'x_2'$. Otherwise $S(T_1') \cap S(T_2')$ would contain a nontrivial segment of c_1' or γ_1', which implies $c_1' \cap c_3'$ or $\gamma_1' \cap \gamma_2'$ contains a nontrivial interval, contradicting to the fact that c_1', c_3' are at different sides and γ_1', γ_2' are also at different sides. It follows that c_1' and c_3' make an angle $\pi/4$. Let $c_4 \subset A$ be the fourth Type I geodesic through v. Also let $D_i \subset A^{[2]}$ ($i = 3, 4$) be the 2-simplex incident to v, lying between c_i and c_{i+1} ($i+1 \mod 4$), and sharing an edge with D_{i-1}. Let $\gamma_i \subset A$, $i = 3, 4$, be the geodesic containing the third edge of D_i. Denote $x_4 = c_4 \cap \gamma_4$, $x_5 = c_1 \cap \gamma_4$. Notice $v', x_1', x_5' \in c_1'$ and $d(v', x_1') = d(v', x_5')$.

Let $T_i' = (v', x_i', x_{i+1}')$ ($i = 3, 4$). Note each T_i' has angle $\pi/4$ at v', and $v'x_{i+1}' \subset S(T_i') \cap S(T_{i+1}')$. Since c_1' and c_3' make an angle $\pi/4$, the triangle inequality implies $\angle_{v'}(x_1', x_5') \leq 3\pi/4$. Hence $x_1' = x_5'$. If $S(T_2') \cap S(T_3') = v'x_3'$, then

$$(S(T_1') - S(T_1') \cap S(T_2')) \cup (S(T_2') - S(T_1') \cap S(T_2')) \cup S(T_3')$$

gives rise to an injective path in Link(Δ_2, v') with length $\pi/2$ from the initial direction of $v'x_1'$ to that of $v'x_4'$. On the other hand, $\angle_{v'}(x_4', x_5') = \pi/4$ and $x_1' = x_5'$, a contradiction. Hence $S(T_2') \cap S(T_3') \neq v'x_3'$. We have proved the implication: if $S(T_1') \cap S(T_2')$ is not a segment, then $S(T_2') \cap S(T_3')$ is not a segment. By working with the 2-simplices of $A^{[2]}$ around x_2 instead of v and using gallery connectedness one concludes that for any two 2-simplices $E_1, E_2 \subset A^{[2]}$ sharing an edge, the intersection image$(g_{E_1}) \cap$ image(g_{E_2}) is not a segment. The above argument also shows that for any Type I geodesic $c \subset A$, and any three consecutive vertices $x, y, z \in c$ with $m(x) = m(y) = m(z) = 8$, we have $x' = z'$. Consequently, for any Type I geodesic $c \subset A$, there are infinitely many $v_i \in c \cap A^{[0]}$ ($i \geq 1$) such that the family $\{\gamma' : \gamma \subset A \text{ is Type I and } \gamma \ni v_i \text{ for some } i\}$ have nonempty intersection. We get a contradiction as in the proof of Proposition 7.12.

□

Corollary 9.19. *Let A be an apartment in Δ_1. Then for any vertex $v \in A$, the geodesics in $\mathcal{D}'_{A,v}$ intersect in a unique vertex of Δ_2.*

Proof. Since $g_A : A \to A'$ is an isomorphism, for any vertex $v \in A$, the g_A-images of the geodesics in $\mathcal{D}_{A,v}$ intersect in a unique vertex. Hence it suffices to show that for any $\xi_1\xi_2 \subset A^{(1)}$, we have $g_A(\xi_i) = \xi_i'$ ($i = 1, 2$), where $g_A : \partial A \to \partial A'$ also denotes the boundary map of $g_A : A \to A'$. By the definition of g_A this claim holds for all Type I geodesics in A. For an arbitrary geodesic $\xi_+\xi_- \subset A^{(1)}$,

one can find a sequence of Type I geodesics $\eta_i\omega_i \subset A^{(1)}$, $i \in \mathbb{Z}$, such that $\xi_+\xi_- \cap \eta_i\omega_i \neq \phi$, $\eta_i\omega_i \to \xi_+$ as $i \to +\infty$ and $\eta_i\omega_i \to \xi_-$ as $i \to -\infty$. It follows that $\eta_i \to \xi_+$ as $i \to +\infty$ and $\eta_i \to \xi_-$ as $i \to -\infty$. Since $h : \partial\Delta_1 \to \partial\Delta_2$ is a homeomorphism, we have $\xi'_+ = h(\xi_+) = h(\lim_{i\to+\infty}\eta_i) = \lim_{i\to+\infty} h(\eta_i) = \lim_{i\to+\infty}\eta'_i = \lim_{i\to+\infty} g_A(\eta_i) = g_A(\xi_+)$. Similarly $\xi'_- = g_A(\xi_-)$ and we are done. $\square$

9.4. *Geodesics through a fixed vertex.* In this section we show that for any vertex $v \in \Delta_1$, the geodesics in $\mathcal{D}'_v$ intersect in a unique vertex.

By Corollary 9.19 for any vertex $v \in \Delta_1$ and any apartment A containing v, the geodesics in $\mathcal{D}'_{A,v}$ intersect in a unique vertex v_A of Δ_2. The proof of Corollary 9.19 actually shows $v_A = g_A(v)$, where $g_A : A \to A'$ is the isomorphism in Proposition 9.18.

Proposition 9.20. *Let Δ_1 and Δ_2 be two Fuchsian buildings with chamber $(2,3,8)$, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. Then for any vertex $v \in \Delta_1$ with $m(v) = 8$, the geodesics in $\mathcal{D}'_v$ intersect in a unique vertex of Δ_2.*

Proof. By Lemma 7.3, we only need to show that $v_{A_1} = v_{A_2}$ holds for any two apartments $A_1, A_2 \subset \Delta_1$ containing v with $\mathrm{Link}(A_1, v)\cap\mathrm{Link}(A_2, v)$ a half apartment in $\mathrm{Link}(\Delta_1, v)$. Let $B_i = \mathrm{Link}(A_i, v)$ $(i = 1, 2)$, and ω_1, ω_2 the two endpoints of $B_1 \cap B_2$. Denote by m the midpoint of $B_1 \cap B_2$, and $m_i \in B_i$ the point in B_i opposite to m. Since $m(v) = 8$, m_1, m_2 and m are vertices. Notice m_1 and m_2 are opposite to each other. Let $\xi_1, \xi_3 \in \partial A_1$ and $\xi_2, \xi_4 \in \partial A_2$ such that $v\xi_1$, $v\xi_2$ have initial direction ω_1 and $v\xi_3$, $v\xi_4$ have initial direction ω_2. Similarly let $\eta_1, \eta_3 \in \partial A_1$ and $\eta_2, \eta_4 \in \partial A_2$ such that $v\eta_3$, $v\eta_4$ have initial direction m, and $v\eta_i$ $(i = 1, 2)$ has initial direction m_i. Then we have $\eta_i\eta_j = v\eta_i \cup v\eta_j$ for $i \neq j$, $\{i, j\} \neq \{3, 4\}$. Lemma 9.4 implies $\xi_1\xi_3$ and $\eta_i\eta_j$ $(1 \le i \neq j \le 4$, $\{i, j\} \neq \{3, 4\})$ are at different sides, and $\xi_2\xi_4$ and $\eta_i\eta_j$ are also at different sides.

We claim the three geodesics $\eta'_1\eta'_2$, $\eta'_1\eta'_3$, $\eta'_3\eta'_2$ have nonempty intersection. The rest of the proof is the same as in Proposition 8.16. Suppose the claim is not true. Then Lemma 7.5 implies there are vertices v_1, v_2, v_3 such that $\eta'_1\eta'_2 \cap \eta'_1\eta'_3 = v_1\eta'_1$, $\eta'_2\eta'_1 \cap \eta'_2\eta'_3 = v_2\eta'_2$, $\eta'_1\eta'_3 \cap \eta'_2\eta'_3 = v_3\eta'_3$; furthermore $T := (v_1, v_2, v_3)$ has three even angles and $S(T)$ is the union of 6 chambers, as shown in Figure 3(h). In this case, $\xi'_1\xi'_3$ and $\eta'_i\eta'_j$ $(1 \le i \neq j \le 3)$ are Type I and are at different sides. Proposition 9.5 implies that $\xi'_1\xi'_3 \cap \eta'_i\eta'_j$ is a vertex for $1 \le i \neq j \le 3$. Denote $w_i = \xi'_1\xi'_3 \cap \eta'_{i+1}\eta'_{i+2}$ $(i \mod 3)$. Since $\xi'_1\xi'_3$ and $\eta'_i\eta'_j$ make even angles and are at different sides, $m(w_i) \neq 2$. Hence $w_i \in v_{i+1}\eta'_{i+1}$ or $w_i \in v_{i+2}\eta'_{i+2}$. We may assume $w_1 \in v_3\eta'_3$. If $w_3 \in v_1\eta'_1$, then $\xi'_1\xi'_3 \cap \eta'_1\eta'_3 \supset w_1w_3$ is not a vertex, contradiction. Similarly we obtain a contradiction if $w_3 \in v_2\eta'_2$.

$\square$

Proposition 9.21. *Let Δ_1 and Δ_2 be two Fuchsian buildings with chamber $(2,3,8)$, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. Then for any vertex $v \in \Delta_1$ with $m(v) = 3$, the geodesics in $\mathcal{D}'_v$ intersect in a unique vertex of Δ_2.*

Proof. Let $v \in \Delta_1$ be a vertex with $m(v) = 3$. We only need to show that $g_{A_1}(v) = g_{A_2}(v)$ holds for any two apartments $A_1, A_2 \subset \Delta_1$ containing v with $\mathrm{Link}(A_1, v) \cap \mathrm{Link}(A_2, v)$ a half apartment in $\mathrm{Link}(\Delta_1, v)$. Let $B_i = \mathrm{Link}(A_i, v)$ $(i = 1, 2)$, and a, b the two endpoints of $B_1 \cap B_2$. There are two edges $e_1 = vv_1$ and $e_2 = vv_2$ that have initial directions a and b respectively. Note $\{m(v_1), m(v_2)\} = \{2, 8\}$. We may assume $m(v_1) = 8$ and $m(v_2) = 2$. Let c, d be the other two vertices on $B_1 \cap B_2$ such that c lies between a and d. Let vv_3, vv_4 be the edges that have initial directions c and d respectively. Then $m(v_4) = 8$, and $T_3 = (v, v_1, v_3)$, $T_4 = (v, v_3, v_4)$ are (boundaries of) chambers. Let $X = S(T_3) \cup S(T_4)$. Notice X is a convex subcomplex of Δ_1, and $X \subset A_1 \cap A_2$.

Since $g_{A_i} : A_i \to A'_i$ is an isomorphism, $g_{A_i}(X) \subset \Delta_2$ is a convex subcomplex. Proposition 9.20 and the fact $m(v_1) = m(v_4) = 8$ imply that $g_{A_1}(v_1) = g_{A_2}(v_1)$ and $g_{A_1}(v_4) = g_{A_2}(v_4)$. As v_3 is the midpoint of v_1v_4, we also have $g_{A_1}(v_3) = g_{A_2}(v_3)$. Set $v'_1 = g_{A_1}(v_1)$, $v'_3 = g_{A_1}(v_3)$, $v'_4 = g_{A_1}(v_4)$. Hence the intersection of the two convex subcomplexes $g_{A_1}(X), g_{A_2}(X) \subset \Delta_2$ contains $v'_1v'_4$. It is

easy to see that either $g_{A_1}(X) \cap g_{A_2}(X) = v_1' v_4'$ or $g_{A_1}(X) = g_{A_2}(X)$. If $g_{A_1}(X) = g_{A_2}(X)$, then $g_{A_1}(v) = g_{A_2}(v)$ and we are done. We shall show that $g_{A_1}(X) \cap g_{A_2}(X) = v_1' v_4'$ does not hold.

Let $\xi_i \in \partial A_i$ $(i = 1, 2)$ be the point such that the initial direction of $v\xi_i$ is opposite to c (the initial direction of vv_3). Then $\angle_v(\xi_1, \xi_2) = 2\pi/3$. Notice $\xi_1\xi_2 \not\subset \Delta_1^{(1)}$, otherwise there would be a triangle with three even angles and one of the angles $= 2\pi/3$, contradicting to Proposition 5.2.

Suppose $g_{A_1}(X) \cap g_{A_2}(X) = v_1' v_4'$. Since $m(v_3') = m(v_3) = 2$, $\angle_{v_3'}(g_{A_1}(v), g_{A_2}(v)) = \pi$. Hence $v_3'\xi_1' \cup v_3'\xi_2' = v_3' g_{A_1}(\xi_1) \cup v_3' g_{A_2}(\xi_2) \subset \Delta_2^{(1)}$ is the geodesic from ξ_1' to ξ_2'. It follows that $\xi_1\xi_2 \subset \Delta_1^{(1)}$, contradicting to the preceding paragraph.

□

Proposition 9.22. *Let Δ_1 and Δ_2 be two Fuchsian buildings with chamber $(2,3,8)$, and $h : \partial\Delta_1 \to \partial\Delta_2$ a homeomorphism that preserves the combinatorial cross ratio almost everywhere. Then for any vertex $v \in \Delta_1$ with $m(v) = 2$, the geodesics in $\mathcal{D}_v'$ intersect in a unique vertex of Δ_2.*

Proof. Let $A_1, A_2 \subset \Delta_1$ be two apartments containing v with $\mathrm{Link}(A_1, v) \cap \mathrm{Link}(A_2, v)$ a half apartment in $\mathrm{Link}(\Delta_1, v)$. Let $B_i = \mathrm{Link}(A_i, v) (i = 1, 2)$, and a, b the two endpoints of $B_1 \cap B_2$. There are two edges $e_1 = vv_1, e_2 = vv_2 \subset A_1 \cap A_2$ that have initial directions a and b at v respectively. Note $m(v_1) = m(v_2) = 3$ or 8. By Propositions 9.20, 9.21, $g_{A_1}(v_1) = g_{A_2}(v_1)$ and $g_{A_1}(v_2) = g_{A_2}(v_2)$. Since $g_{A_i}(v)$ is the only vertex in the interior of $g_{A_i}(v_1) g_{A_i}(v_2)$, we have $g_{A_1}(v) = g_{A_2}(v)$.

□

10. **Triangles and quadrilaterals.** In this section we prove Propositions 5.1, 5.2 and 5.3. We have defined triangles and quadrilaterals in Section 5.

10.1. *Support sets of triangles and quadrilaterals.* Let X be a locally finite CAT(0) 2-complex, e.g., a Fuchsian building. For each $x \in X$, let $\log_x : X - \{x\} \to Link(X, x)$ be the map that sends each $y \neq x$ to the initial direction of xy at x.

Definition 10.1. Let $c \subset X$ be a subset that is homeomorphic to a circle. The *support set* $\mathrm{supp}(c)$ of c is the set of $x \in X - c$ such that $\log_x(c)$ represents a nontrivial class in $H_1(\mathrm{Link}(X, x))$.

The goal of this section is to show that a quadrilateral homeomorphic to a circle must bound a compact surface in X. The triangle case was proved in [X].

Proposition 10.2. ([X]) *Let X be a locally finite CAT(0) 2-complex and $c \subset X$ a triangle homeomorphic to a circle. Then $\overline{supp(c)}$ is a compact surface with boundary c and $\overline{supp(c)} = supp(c) \cup c$.*

Given a triangle or quadrilateral c, we subdivide the 2-complex X such that c becomes part of the 1-skeleton of X. Thus if c is a triangle homeomorphic to a circle and x, y, z its corners, then the segment from $\log_x(y)$ to $\log_x(z)$ is an edge path in the link $\mathrm{Link}(X, x)$. Note $\angle_x(y, z) < \pi$ since c is homeomorphic to a circle. Each edge in $\log_x(y) \log_x(z)$ corresponds to a 2-cell incident to x. Let $\underline{U(c, x)}$ be the union of these 2-cells. The proof in [X] shows that $U(c, x)$ is a neighborhood of x in $\overline{\mathrm{supp}(c)}$.

Lemma 10.3. *Let $c \subset X$ be a triangle or quadrilateral homeomorphic to a circle, and $x \in X - c$. Then $x \in supp(c)$ if and only if $\log_x(c)$ is homotopic to a simple loop with length $< 4\pi$ in $Link(X, x)$.*

Proof. One direction is clear since $\mathrm{Link}(X, x)$ is a finite graph. For the other direction let $x \in \mathrm{supp}(c)$. Then $\log_x(c)$ is homotopically nontrivial in the finite CAT(1) graph $\mathrm{Link}(X, x)$. Notice if x_1x_2 is a geodesic segment which does not contain x, then $\log_x(x_1) \log_x(x_2)$ has length $< \pi$ and $\log_x(x_1x_2)$ is homotopic to $\log_x(x_1) \log_x(x_2)$. It follows that $\log_x(c)$ is homotopic to a closed path with length $< 4\pi$. The unique closed geodesic $\sigma \subset Link(X, x)$ in the free homotopy class of $\log_x(c)$ has the shortest length in the class. It follows that $\mathrm{length}(\sigma) < 4\pi$. Since a closed geodesic in a CAT(1) space has length at least 2π, σ must be a simple closed geodesic.

□

Lemma 10.3 and the arguments in [X] imply the following:

Lemma 10.4. *Let c be a quadrilateral in X that is homeomorphic to a circle. Then $supp(c)$ is a topological surface, $\overline{supp(c)}$ is compact and $\overline{supp(c)} \subset c \cup supp(c)$.*

Now we are ready to prove

Proposition 10.5. *Let X be a locally finite $CAT(0)$ 2-complex and $c \subset X$ a quadrilateral homeomorphic to a circle. Then $\overline{supp(c)}$ is a compact surface with boundary c and $\overline{supp(c)} = supp(c) \cup c$.*

Proof. We shall show that each point $p \in c$ has a neighborhood U such that $U \cap \overline{\mathrm{supp}(c)}$ is homeomorphic to a neighborhood of the origin in the closed upper half plane. Let $x, y, z, w \in c$ be the 4 corners of c in cyclic order. Let $c_1 = xy \cup yz \cup zx$ and $c_2 = wx \cup xz \cup zw$. We first consider the case when $p \in c$ is one of corners, say, $p = w$. We may assume $w \notin c_1$, otherwise c is a triangle and the proposition follows from Proposition 10.2. We orient c_1 and c_2 so that they have opposite orientations on xz. We also orient c so that c is homotopic to $c_1 * c_2$. First suppose $w \notin \mathrm{supp}(c_1)$. By Lemma 10.4 and the remark before Lemma 10.3 there is a neighborhood U of w in X such that $U \cap \overline{\mathrm{supp}(c_2)} = U(c_2, w) \cap U$ and $U \cap \overline{\mathrm{supp}(c_1)} = \phi$. Since c is homotopic to $c_1 * c_2$, it follows from the definition of support set that $U \cap \overline{\mathrm{supp}(c)} = U(c_2, w) \cap U$.

Now suppose $w \in \mathrm{supp}(c_1)$. Recall (see [X]) that the closed path $\log_w(c_1)$ is homotopic to a simple loop $c_{1w} \subset \mathrm{Link}(X, w)$ with length $< 3\pi$. The simple loop c_{1w} can be constructed as follows. The three segments $\log_w(x)\log_w(y)$, $\log_w(y)\log_w(z)$, $\log_w(z)\log_w(x)$ all have length $< \pi$ and their intersections are segments (possibly degenerate): there are $a_0, b_0, c_0 \in \mathrm{Link}(X, w)$ with

$$\log_w(x)\log_w(y) \cap \log_w(x)\log_w(z) = \log_w(x)a_0,$$

$$\log_w(y)\log_w(x) \cap \log_w(y)\log_w(z) = \log_w(y)b_0$$

and

$$\log_w(z)\log_w(y) \cap \log_w(z)\log_w(x) = \log_w(z)c_0.$$

Then $c_{1w} = a_0b_0 \cup b_0c_0 \cup c_0a_0$. Let $U(c_1, w)$ be the union of the 2-cells of X that correspond to the edges in c_{1w}. Then $U(c_1, w)$ is a neighborhood of w in $\mathrm{supp}(c_1)$. Since c is homotopic to $c_1 * c_2$, for a small neighborhood U of w in X we have

$$\overline{\mathrm{supp}(c)} \cap U \subset (U(c_1, w) \cup U(c_2, w)) \cap U$$

and

$$\overline{(U(c_2, w) - U(c_1, w)) \cup (U(c_1, w) - U(c_2, w))} \cap U \subset \overline{\mathrm{supp}(c)} \cap U.$$

We observe that the open 2-cells in $U(c_1, w) \cap U(c_2, w)$ are disjoint from $\mathrm{supp}(c)$: Let p be a point in some open 2-cell of $U(c_1, w) \cap U(c_2, w)$; by the above remark $\log_p(c_1)$ and $\log_p(c_2)$ are homotopic to the circle $\mathrm{Link}(X, p)$ with length 2π; since c is homotopic to $c_1 * c_2$, the path $\log_p(c)$ is either null-homotopic or homotopic to twice of a generate in $\pi_1(\mathrm{Link}(X, p))$; the fact that $\log_p(c)$ has length $< 4\pi$ implies $\log_p(c)$ is null-homotopic. It follows that $\overline{\mathrm{supp}(c)} \cap U = \overline{(U(c_2, w) - U(c_1, w)) \cup (U(c_1, w) - U(c_2, w))} \cap U$. Note $\overline{(U(c_2, w) - U(c_1, w)) \cup (U(c_1, w) - U(c_2, w))}$ is a closed disk and it is the union of the 2-cells corresponding to the edges in the geodesic $\log_w(x)b_0 \cup b_0 \log_w(z) \subset \mathrm{Link}(X, w)$. We also observe that $\log_w(x)b_0 \cup b_0 \log_w(z)$ has length $< 2\pi$.

We next consider the case when p is not a corner. We may assume $p \in \mathrm{interior}(zw)$. Then one of the following occurs:
(1) $pw \cap px = pq$ is a nontrivial segment;
(2) $pw \cap px = \{p\}$ and $px \cap yz$ contains a point r;
(3) $pw \cap px = \{p\}$ and $px \cap yz = \phi$.

Case (1): In this case $zx = zq \cup qx$. Then $T_1 = xw \cup wq \cup qx$ and $T_2 = xy \cup yz \cup zx$ are two triangles, and c is homotopic to $T_1 * T_2$ with suitable orientations. Note $p \notin \mathrm{supp}(T_1)$ since $\log_p(T_1)$ is homotopic to a path with length $< \pi$. It follows that there is a small neighborhood U of p in X

such that $U \cap \overline{\mathrm{supp}(c)} = U \cap \overline{\mathrm{supp}(T_2)}$.

Case (2): Let $T_1 = (z, p, r)$, $T_2 = (r, y, x)$ and $T_3 = (p, x, w)$. Then c is homotopic to $T_1 * T_2 * T_3$ with suitable orientations. Note $p \notin \mathrm{supp}(T_2)$ since $r \in px$ implies the path $\log_p(T_2)$ is homotopic to a path with length $< 2\pi$. The segments $\log_p(r)\log_p(z)$ and $\log_p(x)\log_p(w)$ in $\mathrm{Link}(X, p)$ intersect in a (possibly degenerate) segment $\log_p(r)a$ ($a \in \mathrm{Link}(X, p)$). The path $\log_p(z)a \cup a\log_p(w)$ is a geodesic in $\mathrm{Link}(X, p)$. Th remark before Lemma 10.3 implies $U(T_1, p)$ and $U(T_3, p)$ are neighborhoods of p in $\overline{\mathrm{supp}(T_1)}$ and $\overline{\mathrm{supp}(T_3)}$ respectively. The argument in the second paragraph now implies that the closed disk $\overline{(U(T_1, p) - U(T_3, p))} \cup \overline{(U(T_3, p) - U(T_1, p))}$ is a neighborhood of p in $\overline{\mathrm{supp}(c)}$, and that the disk is the union of the 2-cells that correspond to the edges in $\log_p(z)a \cup a\log_p(w)$.

Case (3): In this case $T := (p, x, w)$ and $Q := (z, y, x, p)$ are homeomorphic to a circle. The path c is homotopic to $T * Q$ with suitable orientations. The discussion in the second paragraph implies $\overline{\mathrm{supp}(Q)}$ determines a geodesic segment $Q_p \subset \mathrm{Link}(X, p)$ with length $< 2\pi$, and $U(Q, p)$ is a neighborhood of p in $\overline{\mathrm{supp}(Q)}$, where $U(Q, p)$ is the union of the 2-cells that correspond to the edges in Q_p. On the other hand, $U(T, p)$ is a neighborhood of p in $\overline{\mathrm{supp}(T)}$, and the link of $U(T, p)$ at p is the segment $\log_p(x)\log_p(w)$ which has length $< \pi$. Since $\mathrm{Link}(X, p)$ is a finite CAT(1) graph and the distance in $\mathrm{Link}(X, p)$ between $\log_p(z)$ and $\log_p(w)$ is at least π, the intersection $Q_p \cap \log_p(x)\log_p(w) = \log_p(x)a$ is a (possibly degenerate) segment. Let $U(c, p)$ be the union of the 2-cells that correspond to the edges in $(Q_p - \log_p(x)a) \cup a\log_p(w)$. Then $U(c, p)$ is a disk. The argument in the second paragraph shows that $U(c, p)$ is a neighborhood of p in $\overline{\mathrm{supp}(c)}$.

□

The following lemma is a special case of a general observation due to B. Kleiner.

Lemma 10.6. *Let $c \subset X$ be a triangle or quadrilateral homeomorphic to a circle, and $x \in \mathrm{supp}(c)$. Then any nontrivial geodesic segment yx in X can be extended into $\mathrm{supp}(c)$, that is, there is some point $z \in X - \{x\}$ such that $xz \subset \mathrm{supp}(c)$ and $x \in yz$.*

Proof. Let $\xi \in \mathrm{Link}(X, x)$ be the initial direction of yx at x. Since $x \in \mathrm{supp}(c)$, $\log_x(c)$ is homotopic to a nontrivial loop c_x in the CAT(1) graph $\mathrm{Link}(X, x)$ and $\mathrm{interior}(U(c, x)) \subset \mathrm{supp}(c)$, where $U(c, x)$ is the union of 2-cells corresponding to the edges in c_x. It follows that there exists some $\eta \in c_x$ such that the distance in $\mathrm{Link}(X, x)$ from ξ to η is π. Now we choose a geodesic segment $xz \subset \mathrm{interior}(U(c, x))$ such that the initial direction of xz at x is η. Then $yz = yx \cup xz$ is an extention of yx into $\mathrm{supp}(c)$.

□

10.2. *Gauss-Bonnet formula for piecewise Riemannian 2-complexes.* In this section we recall the Gauss-Bonnet formula for finite 2-complexes with piecewise Riemannian metrics (see for example [BB]).

Let X be a finite 2-complex. We suppose each 2-cell of X has a Riemannian metric such that the 1-cells are geodesics, and for any two 2-cells f_1, f_2 with $f_1 \cap f_2 \neq \phi$, the induced metrics on $f_1 \cap f_2$ from f_1 and f_2 coincide.

For a vertex v let

$$\chi(v) = \chi(\mathrm{Link}(X, v)),$$

the Euler characteristic of the link $\mathrm{Link}(X, v)$. For a 2-cell f incident to v, denote by $\alpha(v, f)$ the interior angle of f at v. The complete angle at v is

$$\alpha(v) = \Sigma\alpha(v, f),$$

where f varies over all 2-cells incident to v. The curvature measure of v is then by definition

$$\kappa(v) = (2 - \chi(v))\pi - \alpha(v).$$

For a 2-cell f denote by K the Gaussian curvature of f. Then the curvature of f by definition is

$$\kappa(f) = \int_f K.$$

Now the total curvature of X is defined by:

$$\kappa(X) = \Sigma_s \kappa(s)$$

where s varies over all the 0-cells and 2-cells of X (the curvatures of the 1-cells are 0 since they are geodesics).

The Gauss-Bonnet formula (see [BB]) says

$$\kappa(X) = 2\pi\chi(X).$$

Note $\chi(X) = 1$ when X is homeomorphic to a disk. If a 2-cell f has constant Gaussian curvature -1, then $\kappa(f) = -\text{Area}(f)$.

Let Δ be a Fuchsian building. For a finite sequence of pairwise distinct points $P = < x_1, \cdots, x_l >$ $(l \geq 3)$ in Δ, define $d(P) = (l-2)\pi - \Sigma_{i=1}^{l} \angle_{x_i}(x_{i-1}, x_{i+1})$ $(i \mod l)$. If $T = (x, y, z)$ is a triangle, set $d(T) = d(< x, y, z >)$. Similarly we define $d(Q)$ if Q is a quadrilateral. For a triangle or quadrilateral $c \subset \Delta^{(1)}$ homeomorphic to a circle, let $n(c)$ be the number of chambers in $\overline{\text{supp}(c)}$. Recall $\overline{\text{supp}(c)}$ is a finite subcomplex. We also denote by A_0 the area of a chamber.

Corollary 10.7. *Let Δ be a Fuchsian building, and $c \subset \Delta^{(1)}$ a triangle or quadrilateral homeomorphic to a circle. If $\overline{supp(c)}$ is homeomorphic to a closed disk, then $d(c) \geq n(c)A_0$.*

Proof. Let $X = \overline{\text{supp}(c)}$. Since Δ is CAT(-1), $\kappa(v) \leq 0$ if $v \in X$ is a vertex but not a corner of c. On the other hand, if v is a corner of c, then the length of $\text{Link}(X, v)$ is larger than or equal to the angle of c at v. The corollary follows from these two observations and Gauss-Bonnet formula.

□

Let us make some observation about A_0. The Gauss-Bonnet formula applied to the chamber R implies: $A_0 = (k-2)\pi - \Sigma(\partial R)$, where R is a k-gon and $\Sigma(\partial R)$ is the sum of angles at the vertices of R.

- if $k \geq 5$, then $A_0 \geq \pi/2$, with equality precisely when $k = 5$ and all the angles of R equal to $\pi/2$;
- if $k = 4$, then $A_0 \geq \pi/6$, with equality precisely when R has angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}$;
- if $k = 3$ and R has no right angle, then $A_0 \geq \pi/12$, with equality precisely when $R = (3, 3, 4)$;
- if $R = (2, 8, 8)$, then $A_0 = \pi/4$;
- if $R = (2, 6, 8)$, then $A_0 = 5\pi/24$;
- if $R = (2, 6, 6)$, then $A_0 = \pi/6$;
- if $R = (2, 4, 6)$, then $A_0 = \pi/12$;
- if $R = (2, 4, 8)$, then $A_0 = \pi/8$;
- if $R = (2, 3, 8)$, then $A_0 = \pi/24$.

10.3. *Triangles in the 1-skeleton.* Let Δ be a Fuchsian building and $c \subset \Delta^{(1)}$ a triangle or quadrilateral homeomorphic to a circle. Then $X := \overline{\text{supp}(c)}$ is a finite subcomplex homeomorphic to a compact surface with boundary. Notice a vertex $v \in X$ is a special point if and only if $\kappa(v) < 0$. The remark before Lemma 10.3 implies that the corners of a triangle are not special points. Proposition 5.1 follows from the following result:

Proposition 10.8. *Let Δ be a Fuchsian building and $T \subset \Delta^{(1)}$ a triangle that is homeomorphic to a circle. Then $\overline{supp(T)}$ contains no special points. In particular, $\overline{supp(T)}$ is homeomorphic to a closed disk and is convex in Δ.*

The second claim in the proposition follows easily from the first one: assume $X = \overline{\text{supp}(T)}$ contains no special points; then X is locally convex in the CAT(-1) space Δ; it follows that X is actually convex in Δ and hence is contractible.

Remark 10.9. I suspect that any triangle or quadrilateral homeomorphic to a circle in a CAT(0) 2-complex must bound a disk, at least for those CAT(0) 2-complexes that admit cocompact groups of isometries.

Let $T = (x, y, z) \subset \Delta^{(1)}$ be a triangle homeomorphic to a circle. Recall the angles of the chamber R lie in $\{\pi/2, \pi/3, \pi/4, \pi/6, \pi/8\}$. In particular, they are integral multiples of $\pi/24$. It follows that $d(T)$ is an integral multiple of $\pi/24$. On the other hand, $d(T) > 0$ because Δ is a CAT(-1) space. Therefore $\pi/24 \leq d(T) \leq 5\pi/8$. We will prove Proposition 10.8 by inducting on $d(T)$, starting with triangles with $d(T) = \pi/24$. We first make an observation that shall be used often later.

The following lemma follows easily from the triangle inequality and the fact that $d(P) > 0$ for any finite sequence of pairwise distinct points $P =< x_1, \cdots, x_l > (l \geq 3)$.

Lemma 10.10. *Let $P =< x_1, \cdots, x_l > (l \geq 3)$ be a sequence of pairwise distinct points, and $y_i \in x_i x_{i+1}$, $y_j \in x_j x_{j+1}$ with $i < j$. If $P' =< y_i, x_{i+1}, \cdots, x_j, y_j >$ is a sequence of pairwise distinct points, then $d(P') < d(P)$.*

We first study triangles with $d(T) = \pi/24$.

Lemma 10.11. *Let Δ be a Fuchsian building and $T = (x, y, z) \subset \Delta^{(1)}$ a triangle that is homeomorphic to a circle. If $d(T) = \pi/24$, then $R = (2, 3, 8)$ and T is the boundary of some chamber.*

Proof. We claim that none of the segments xy, yz and zx contains any vertex in the interior. Suppose, say, xy contains a vertex p in the interior. Then there is an edge $pq \subset \overline{\text{supp}(T)}$ such that interior$(pq) \subset \text{supp}(T)$. Since $\overline{\text{supp}(T)}$ is a compact surface, Lemma 10.6 implies pq extends into supp(T) and the extension hits the boundary of $\overline{\text{supp}(T)}$ (which is T) at some point r. The segment pr lies in $\Delta^{(1)}$. Let c' be the union of pr with one of the two components of $T - \{p, r\}$. Then c' is either a triangle or a quadrilateral. Lemma 10.10 implies $d(c') < d(T) = \pi/24$, which is a contradiction since $d(c') > 0$ is an integral multiple of $\pi/24$. Hence xy, yz and zx are three edges in Δ. Similarly one shows that $\log_x(y) \log_x(z)$ is an edge in Link(Δ, x). It follows that xy and xz lie in the boundary of some chamber C and we conclude that C is actually a triangle and T is its boundary. Since $d(T) = \pi/24$, $A_0 = \pi/24$. The observation about A_0 shows $R = (2, 3, 8)$.

□

For a triangle or quadrilateral $c \subset \Delta^{(1)}$ homeomorphic to a circle and $x \in \overline{\text{supp}(c)}$, let c_x be the link Link$(\overline{\text{supp}(c)}, x)$. By Propositions 10.2 and 10.5 c_x is a circle if $x \in \text{supp}(c)$ and is a segment if $x \in c$.

Lemma 10.12. *Let $T = (x, y, z) \subset \Delta^{(1)}$ be a triangle homeomorphic to a circle with $d(T) = \frac{k\pi}{24}$, $2 \leq k \leq 15$. Assume $\overline{\text{supp}(T')}$ has no special points for every triangle $T' \subset \Delta^{(1)}$ homeomorphic to a circle with $d(T') \leq \frac{(k-1)\pi}{24}$. Then T contains no special points of $\overline{\text{supp}(T)}$.*

Proof. Suppose the lemma is false. Since the corners of T are not special points, we may assume there is a special point $p \in \text{interior}(xy)$. Then c_p is a segment with length $> \pi$. Note c_p is an edge path in Link(Δ, p). Let $\eta \in c_p$ be the point such that the subsegment of c_p from $\log_p(x)$ to η has length π. Then η is a vertex in Link(Δ, p). There is an edge $pq \subset \overline{\text{supp}(T)}$ with initial direction η and interior$(pq) \subset \text{supp}(T)$. We extend the geodesic pq into supp(T) until it hits a point $r \in T$. The uniqueness of geodesic implies $r \in \text{interior}(yz)$. Let $T_1 = (x, r, z)$ and $T_2 = (p, y, r)$. Lemma 10.10 implies $d(T_i) \leq \frac{(k-1)\pi}{24}$ for $i = 1, 2$. The assumption then implies that $\overline{\text{supp}(T_i)}$ contains no special points. In particular $\overline{\text{supp}(T_i)}$ is homeomorphic to a closed disk and we can apply Corollary 10.7 to T_i.

Note T_2 has an even angle at p and $A_0 \geq \pi/24$. Since $\angle_r(p, z) + \angle_r(p, y) \geq \pi$, Corollary 10.7 applied to T_2 shows that $\angle_r(p, z) \geq \angle_p(r, y) + \angle_y(x, z) + 2A_0 \geq 2 \times \frac{\pi}{m(p)} + \pi/8 + 2 \times \pi/24$. Since there are at least $m(p)$ chambers in $\overline{\text{supp}(T_1)}$ incident to p, Corollary 10.7 applied to T_1 implies that

$\pi \geq \pi/8+\pi/8+\angle_r(p,z)+m(p)A_0$. Combining these two inequalities we obtain $m(p)A_0+\frac{2\pi}{m(p)} \leq \frac{13\pi}{24}$. Since A_0 is an integral multiple of $\pi/24$ and $m(p) \in \{2,3,4,6,8\}$, this inequality never holds and we have a contradiction.

□

Lemma 10.13. *Let $T=(x,y,z) \subset \Delta^{(1)}$ be a triangle homeomorphic to a circle with $d(T)=\frac{k\pi}{24}$, $2 \leq k \leq 15$. Assume $\overline{supp(T')}$ has no special points for every triangle $T' \subset \Delta^{(1)}$ homeomorphic to a circle with $d(T') \leq \frac{(k-1)\pi}{24}$. Then there is no special point in supp(T).*

Proof. Suppose there is a special point $p \in \text{supp}(T)$. Then c_p has length $> 2\pi$. There is at least one edge $pq \subset \overline{\text{supp}(T)}$ such that its extension in supp(T) eventually hits T at a point p_1 different from x, y and z. We may assume $p_1 \in \text{interior}(xy)$. Denote $\eta_1 = \log_p(p_1)$. Let $\eta_2 \neq \eta_3 \in c_p$ be the two points such that their distance in c_p from η_1 is exactly π. We extend p_1p inside supp(T) in two ways, one in the direction η_2, the other in the direction η_3. Suppose they eventually hit T at p_2 and p_3 respectively. Note $p_2, p_3 \in T - xy$.

We claim one of p_2, p_3 lies in xz and the other lies in yz. Suppose otherwise, say, $p_2, p_3 \in yz$ and $p_3 \in \text{interior}(p_2y)$. Let $T_1=(p_1,p_2,y)$. Lemma 10.10 and the assumption imply that $\overline{\text{supp}(T_1)}$ contains no special points. This contradicts to the facts that $p \in p_1p_2, p \in p_1p_3$ and that the initial directions of pp_2 and pp_3 at p are different.

We may assume $p_2 \in xz$ and $p_3 \in yz$. Let $T_2=(p_1,p_2,x)$, $T_3=(p_1,p_3,y)$ and $Q=(p,p_2,z,p_3)$. Also denote $m=m(p)$. Lemma 10.10 and the assumption imply that $\overline{\text{supp}(T_i)}$ $(i=2,3)$ is homeomorphic to a closed disk and we can apply Corollary 10.7 to T_i. Since $p \in \text{interior}(p_1p_i)$, there are at least m chambers in $\overline{\text{supp}(T_i)}$. Now Corollary 10.7 implies $\angle_{p_1}(p_2,x)+\angle_{p_2}(p_1,x)+\angle_x(y,z)+mA_0 \leq \pi$, $\angle_{p_1}(p_3,y)+\angle_{p_3}(p_1,y)+\angle_y(x,z)+mA_0 \leq \pi$. Since Δ is CAT(-1) and all the angles of Q are integral multiples of $\pi/24$, we have $\angle_{p_2}(p,z)+\angle_p(p_2,p_3)+\angle_{p_3}(p,z)+\angle_z(p_2,p_3) \leq 2\pi-\pi/24$. Notice $\angle_{p_2}(p,z)+\angle_{p_2}(p,x) \geq \pi$, $\angle_{p_3}(p,z)+\angle_{p_3}(p,y) \geq \pi$, $\angle_{p_1}(p,y)+\angle_{p_1}(p,x) \geq \pi$. Since the angles of T are $\geq \pi/8$, the above inequalities imply $2\pi/m+2mA_0 \leq 7\pi/12$. The second term here, $2mA_0 \geq 4 \times \pi/24 = \pi/6$. It follows that $2\pi/m \leq 5\pi/12$, which implies $m \geq 6$. The first term $2\pi/m \geq \pi/4$. It follows that $\pi/3 \geq 2mA_0 \geq 12A_0$, or, $A_0 \leq \pi/36$, contradicting to the fact that $A_0 \geq \pi/24$.

□

The proof of Proposition 10.8 is now complete. Next we consider Proposition 5.2.

Lemma 10.14. *Let Δ be a Fuchsian building with chamber R. If R is not a triangle, then no triangle in $\Delta^{(1)}$ is homeomorphic to a circle.*

Proof. Suppose $T \subset \Delta^{(1)}$ is a triangle homeomorphic to a circle. Assume R is a k-gon with $k \geq 4$ and let α_0 be the smallest angle of R. Proposition 10.8 and Corollary 10.7 imply that $3\alpha_0+n(T)A_0 \leq \pi$. Since T is a triangle and R is not, $n(T) \geq 2$. Notice $A_0 \geq \pi/6$ for $k \geq 4$. It follows that $\alpha_0 \leq 2\pi/9$ and so $\alpha_0 = \pi/6$ or $\pi/8$. By Gauss-Bonnet $A_0 \geq \pi/3$. Repeating the above argument we obtain $\alpha_0 \leq \pi/9$, contradicting to the fact that $\alpha_0 \geq \pi/8$.

□

There are two ways to prove the other claims in Proposition 5.2. The first method is to use Corollary 10.7 and the observation about A_0. Basically Corollary 10.7 puts severe restriction on the possible configurations for $\overline{\text{supp}(T)}$ and one can list all the triangles in $\Delta^{(1)}$ that are homeomorphic to a circle. We omit the details here. On the other hand, Proposition 10.8 implies that $\overline{\text{supp}(T)}$ is isomorphic to a convex subcomplex in an apartment. So the second method is to find all the triangles by examining the tessellation of $\mathbb{H}^2$ by the chamber. Here we exhibit the tessellations by $(2,6,8)$, $(2,4,8)$ and $(2,3,8)$.

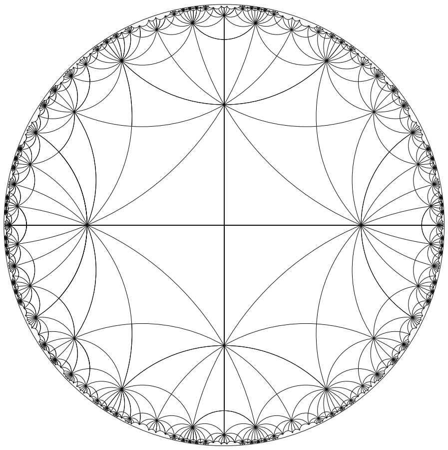

FIGURE 7. Tessellation of $\mathbb{H}^2$ by $(2, 6, 8)$

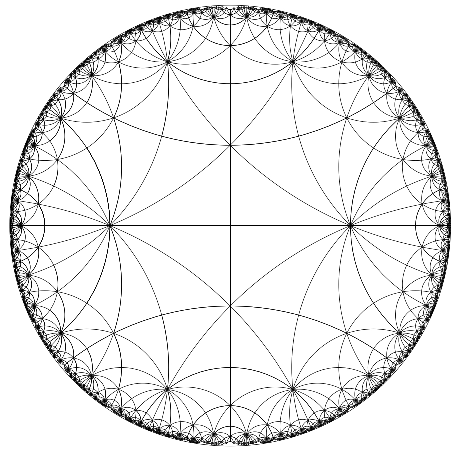

FIGURE 8. Tessellation of $\mathbb{H}^2$ by $(2, 4, 8)$

10.4. *Quadrilaterals must bound disks.* In this section we show that $\overline{\text{supp}(Q)}$ is homeomorphic to a closed disk for any quadrilateral $Q \subset \Delta^{(1)}$ homeomorphic to a circle. Proposition 5.3 then follows from Corollary 10.7. The proof is based on the following lemma.

Lemma 10.15. *Let $Q = (x, y, z, w) \subset \Delta^{(1)}$ be a quadrilateral that is homeomorphic to a circle. Then $\overline{supp(Q)}$ is homeomorphic to a closed disk if the following conditions are satisfied:*
(1) *there is no special point in $supp(Q)$;*
(2) *x, y, z, w are not special points;*
(3) *for each $p \in Q$, the path Q_p has length $\leq 2\pi$;*
(4) *there is at most one special point in the interior of each side of Q.*

Proof. By Proposition 10.5, $\overline{\text{supp}(Q)}$ is a compact surface with boundary Q. It suffices to prove that $\overline{\text{supp}(Q)}$ is simply connected. Notice $\overline{\text{supp}(Q)}$ has nonpositive curvature with respect to the path metric, which is denoted by d'. Suppose $\overline{\text{supp}(Q)}$ is not simply connected. Then there is a simple closed geodesic (with respect to d') $\gamma \subset \overline{\text{supp}(Q)}$ (see p. 202 of [BH]). γ must contain special points since otherwise γ is a closed geodesic in the CAT(-1) space Δ. One also sees that γ contains at least three special points. The assumption implies that there are either 3 or 4 special points on γ, and hence γ is actually a triangle or quadrilateral in Δ with the special points as corners. For any special point $p \in \gamma \cap Q$, condition (3) implies that the angle (measured in Δ) that γ makes at p is at least $\pi/2$. Now we have a contradiction since the angles at the corners of the triangle or quadrilateral γ are at least $\pi/2$ and Δ is a CAT(-1) space.

□

We first show condition (3) of Lemma 10.15 holds:

Lemma 10.16. *Let Δ be a Fuchsian building and $Q = (x_1, x_2, x_3, x_4) \subset \Delta^{(1)}$ a quadrilateral that is homeomorphic to a circle. Then $length(Q_p) \leq 2\pi$ for every $p \in Q$.*

Proof. Suppose there is some $p \in Q$ with length$(Q_p) > 2\pi$. Then p is vertex. The proof of Lemma 10.5 shows that p is not a corner. We may assume $p \in interior(x_1x_2)$. We claim $m(p) \neq 2$. Suppose $m(p) = 2$. Then length$Q_p \geq 3\pi$ and there are two edges $pq_1, pq_2 \subset \overline{\text{supp}(Q)}$ such that the angle between any two of px_1, pq_1, pq_2 and px_2 is π. We extend pq_1, pq_2 inside supp(Q) until they hit Q at

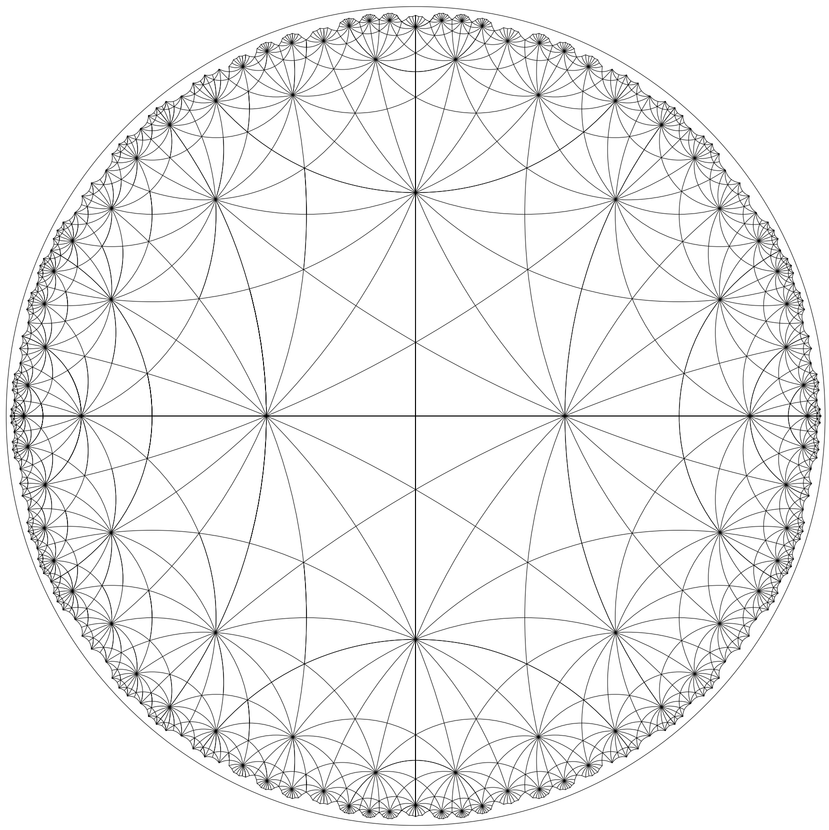

FIGURE 9. Tessellation of $\mathbb{H}^2$ by $(2,3,8)$, created by Martin Deraux

r_1 and r_2 respectively. The uniqueness of geodesic implies $r_1, r_2 \in x_3x_4$. But then $r_1r_2 = r_1p \cup pr_2$ is not contained in x_3x_4, contradicting to the uniqueness of geodesic. Hence $m(p) \geq 3$. Since length$Q_p > 2\pi$, there are edges $pq_1, \cdots, pq_l \subset \overline{\text{supp}(Q)}$, with $l \geq 6$ and interior$(pq_i) \subset \text{supp}(Q)$. We extend each pq_i until it hits Q at some point r_i. We orient $x_1x_4 \cup x_4x_3 \cup x_3x_2$ from x_1 to x_2 and relabel the points $q_1, \cdots, q_l$ such that $r_1, \cdots, r_l$ are in linear order in $x_1x_4 \cup x_4x_3 \cup x_3x_2$ when one travels from x_1 to x_2. We set $r_0 = x_1$ and $r_{l+1} = x_2$. Since $l \geq 6$, there exist at least 3 points r_i, r_{i+1}, r_{i+2} that lie on the same side of Q. In particular, the triangle (p, r_i, r_{i+2}) has a side r_ir_{i+2} containing at least two edges. Proposition 5.2 implies R is a right triangle.

We claim $R = (2,3,8)$. Suppose $R \neq (2,3,8)$. Then $m(p) \geq 4$ and $l \geq 8$. There are 4 points $r_i, r_{i+1}, r_{i+2}, r_{i+3}$ that lie on the same side of Q. Proposition 5.2 applied to (p, r_i, r_{i+1}), (p, r_{i+1}, r_{i+2}) and (p, r_{i+1}, r_{i+3}) shows that $\angle_{r_{i+1}}(r_i, p)$, $\angle_{r_{i+1}}(r_{i+2}, p)$, $\angle_{r_{i+2}}(r_{i+1}, p)$, $\angle_{r_{i+2}}(r_{i+3}, p) \leq \pi/2$. The triangle inequality then implies all these 4 angles are equal to $\pi/2$. In particular, (p, r_{i+1}, r_{i+2}) has two right angles, which is impossible. Hence $R = (2,3,8)$ and $m(p) = 3$ or 8.

Suppose $m(p) = 8$. Since the initial direction of px_1 and px_2 are of the same type, $l \geq 17$. There are 7 points $r_i, \cdots, r_{i+6}$ that lie on the same side of Q. The argument in the preceding paragraph shows at least one of the 4 angles $\angle_{r_{i+2}}(p, r_{i+1})$, $\angle_{r_{i+2}}(p, r_{i+3})$, $\angle_{r_{i+3}}(p, r_{i+2})$, $\angle_{r_{i+3}}(p, r_{i+4})$, is $> \pi/2$. Assume, say, $\angle_{r_{i+2}}(p, r_{i+1}) > \pi/2$. Let $T_1 = (p, r_i, r_{i+1})$ and $T_2 = (p, r_{i+1}, r_{i+2})$. Proposition 5.2

applied to T_2 shows $\angle_{r_{i+1}}(p, r_{i+2}) = \pi/8$. It follows that $\angle_{r_{i+1}}(p, r_i) = 7\pi/8$ and $d(T_1) < 0$, which is impossible.

Finally assume $m(p) = 3$. Then $l \geq 6$. Notice Proposition 5.2 implies if $(y_1, y_2, y_3) \subset \Delta^{(1)}$ is homeomorphic to a circle and $m(y_1) = 3$, then $\angle_{y_2}(y_1, y_3) \leq \pi/2$ and equality holds only when (y_1, y_2, y_3) is the boundary of a chamber. It follows that if r_i, r_{i+1}, r_{i+2} lie on the same side of Q, then $m(r_{i+1}) = 2$ and (p, r_i, r_{i+1}), (p, r_{i+1}, r_{i+2}) are boundaries of chambers. This implies that each side of Q contains at most 3 of the r_i's. We claim there is no i such that $\{r_i, r_{i+1}, r_{i+2}\} \subset x_1x_4 - \{x_4\}$ and $\{r_{i+3}, r_{i+4}, r_{i+5}\} \subset x_3x_4 - \{x_4\}$; and similar claim also holds about the two sides of Q incident to x_3. Suppose there is such an i. Let $Q_1 = (p, r_{i+2}, x_4, r_{i+3})$. Then the observation about angles implies that $d(Q_1) < 0$, which is impossible. Recall there are at least 8 r_j's. If $x_3 = r_i$ for some i, then x_2x_3 contains exactly 3 r_j's, and each of $x_1x_4 - \{x_4\}$, $x_3x_4 - \{x_4\}$ contains exactly 3 r_j's, contradicting to the above claim. Hence x_3 is distinct from all the r_j's. Similarly x_4 is also distinct from all the r_j's. The above claim implies x_3x_4 contains exactly two r_j's and they lie in interior(x_3x_4), and each of $x_1x_4 - \{x_4\}$, $x_2x_3 - \{x_3\}$ also contains exactly 3 r_j's. So we have $r_1, r_2 \in$ interior(x_1x_4), $r_3, r_4 \in$ interior(x_3x_4) and $r_5, r_6 \in$ interior(x_2x_3). Consider $T = (p, r_3, r_4)$. Since $m(p) = 3$, Proposition 5.2 implies at least one of $\angle_{r_3}(p, r_4)$, $\angle_{r_4}(p, r_3)$ is $\leq \pi/4$. We may assume $\angle_{r_4}(p, r_3) \leq \pi/4$. It follows that $\angle_{r_4}(p, x_3) \geq 3\pi/4$ and $d(Q_2) < 0$, which is impossible, where $Q_2 = (p, r_4, x_3, r_5)$. The contradiction proves the lemma.

□

The proof of Proposition 5.3 is similar to that of Proposition 10.8: we induct on $d(Q) = \frac{k\pi}{24}$, starting with $k = 1, 2$.

Lemma 10.17. *Let Δ be a Fuchsian building and $Q = (x_1, x_2, x_3, x_4) \subset \Delta^{(1)}$ a quadrilateral that is homeomorphic to a circle. Then $d(Q) \geq \frac{2\pi}{24}$. Moreover, if $d(Q) = \frac{2\pi}{24}$, then $R = (2, 3, 8)$ and $\overline{supp(Q)}$ can be obtained by gluing two chambers along an edge.*

Proof. Note $d(Q) > 0$ is an integral multiple of $\pi/24$. Hence $d(Q) \geq \frac{\pi}{24}$. Suppose $d(Q) = \frac{\pi}{24}$. Then the proof of Lemma 10.11 shows that x_ix_{i+1} ($i \mod 4$) is an edge and $\log_{x_i}(x_{i-1}) \log_{x_i}(x_{i+1})$ is an edge in Link(Δ, x_i). From this it is not hard to see that Q is the boundary of a chamber. In particular the chamber is a 4-gon. In this case $d(Q) = A_0 \geq \pi/6$, contradicting to the assumption that $d(Q) = \frac{\pi}{24}$. Hence $d(Q) \geq \frac{2\pi}{24}$.

Now suppose $d(Q) = \frac{2\pi}{24}$. Assume some side of Q, say x_1x_2, contains a vertex p in the interior. For an edge $pp' \subset \overline{\text{supp}(Q)}$ with interior$(pp') \subset$ supp(Q), we extend pp' inside supp(Q) until it hits Q at some point q. Since $d(Q') \geq \frac{2\pi}{24}$ for all quadrilaterals, Lemma 10.10 implies $q \in$ interior(x_1x_4), or $q \in$ interior(x_2x_3). We may assume $q \in$ interior(x_1x_4). Denote $T = (p, q, x_1)$ and $P = < p, x_2, x_3, x_4, q >$. Lemma 10.10 implies $d(T) = d(P) = \pi/24$. By Lemma 10.11, $R = (2, 3, 8)$ and T is the boundary of some chamber. At least one of $m(p)$, $m(q)$ is $\neq 2$. We may assume $m(p) \neq 2$. Then there is an edge $pp'' \subset \overline{\text{supp}(Q)}$ different from pp' with interior$(pp'') \subset$ supp(Q). We extend pp'' inside supp(Q) until it hits Q at some point $r \neq q$. Then $r \in$ interior(x_1x_4), or $r \in$ interior(x_2x_3). Note $r \in$ interior(x_1x_4) cannot hold since otherwise both x_1q and x_1r are edges. Now Lemma 10.10 applied to P and $T' = (p, r, x_2)$ implies $d(T') < \pi/24$, a contradiction. Hence each x_ix_{i+1} is an edge.

The first paragraph shows there is some i such that $\log_{x_i}(x_{i-1}) \log_{x_i}(x_{i+1})$ contains at least two edges. It implies that there is some edge $x_ip' \subset \overline{\text{supp}(Q)}$ with interior$(x_ip') \subset$ supp(Q). We extend x_ip' inside supp(Q) until it hits some $q \in Q$. Since $d(Q) = 2\pi/24$ and $d(Q') \geq 2\pi/24$ for all quadrilaterals Q', Lemma 10.10 implies $q = x_{i+2}$. Let $T_1 = (x_i, x_{i+1}, x_{i+2})$ and $T_2 = (x_i, x_{i-1}, x_{i+2})$. By Lemma 10.10, $d(T_1) = d(T_2) = \pi/24$. Lemma 10.11 then implies that $R = (2, 3, 8)$ and both T_1 and T_2 are the boundaries of chambers.

□

We first consider the case when one of the corners of Q is a special point. Recall for each $x \in \overline{\text{supp}(Q)}$, Q_x denotes the link of $\overline{\text{supp}(Q)}$ at x.

Lemma 10.18. *Let Δ be a Fuchsian building and $Q = (x_1, x_2, x_3, x_4) \subset \Delta^{(1)}$ a quadrilateral that is homeomorphic to a circle. If Q_{x_1} has length $> \pi$, then $R = (2, 3, 8)$ and $\overline{supp(Q)}$ must be as shown in Figure 10.*

Proof. Let $\eta \in Q_{x_1}$ be the point such that the subsegment of Q_{x_1} from $\log_{x_1}(x_4)$ to η has length π. Let $x_1 q \subset \overline{\text{supp}(Q)}$ be the edge with initial direction η at x_1. We extend $x_1 q$ inside supp(Q) until it hits Q at some point r. The uniqueness of geodesic implies $r \in$ interior$(x_2 x_3)$. Let $T_1 = (x_4, r, x_3)$ and $T_2 = (x_1, x_2, r)$. Since the side $x_4 r = x_4 x_1 \cup x_1 r$ of T_1 contains at least two edges, Proposition 5.2 implies R is a right triangle and $\angle_r(x_1, x_3) \leq \pi/2$. We claim $\angle_r(x_1, x_2) \leq \pi/2$. Suppose $\angle_r(x_1, x_2) > \pi/2$. Then Proposition 5.2 implies that $R = (2, 3, 8)$ and $\angle_r(x_1, x_2) = 2\pi/3$, $\angle_{x_1}(r, x_2) = \pi/8$. Now consider T_1: the side $x_4 r$ contains the vertex x_1 in the interior with $m(x_1) = 8$ and one endpoint r of $x_4 r$ satisfies $m(r) = 3$. However, by Proposition 5.2 no such triangle exists.

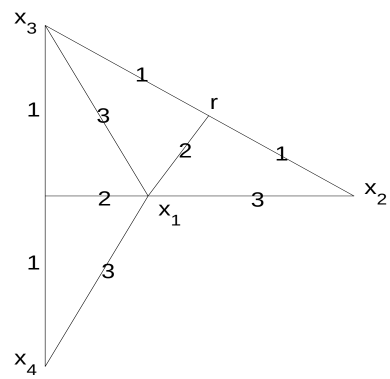

FIGURE 10. Corner x_1 is a special point

Now $\angle_r(x_1, x_3) \leq \pi/2$, $\angle_r(x_1, x_2) \leq \pi/2$ and the triangle inequality imply that $\angle_r(x_1, x_2) = \angle_r(x_1, x_3) = \pi/2$. Consider T_2. Since $\angle_r(x_1, x_2) = \pi/2$, $\angle_{x_1}(r, x_2) < \pi/2$. In particular, $m(x_1) > 2$. Now consider T_1: the angle at r is $\pi/2$, the side $x_4 r$ contains the vertex x_1 in the interior with $m(x_1) > 2$. Proposition 5.2 implies $R = (2, 3, 8)$, $m(r) = 2$, $m(x_1) = 3$ and $\overline{\text{supp}(T_1)}$ is as shown in Figure 3(d). It follows that $\angle_{x_1}(r, x_2) = \pi/3$ and by Proposition 5.2 T_2 is the boundary of a chamber. Now the lemma follows.

□

From now on we assume that none of the 4 corners of Q is a special point.

Lemma 10.19. *Let $Q = (x_1, x_2, x_3, x_4) \subset \Delta^{(1)}$ be a quadrilateral that is homeomorphic to a circle with $d(Q) = \frac{k\pi}{24}$, $k \geq 3$. Suppose $\overline{supp(Q')}$ is homeomorphic to a closed disk for every quadrilateral $Q' \subset \Delta^{(1)}$ homeomorphic to a circle with $d(Q') \leq \frac{(k-1)\pi}{24}$. Then for each i, interior$(x_i x_{i+1})$ (i mod 4) contains at most one special point.*

Proof. Suppose the lemma is not true. We may assume interior$(x_1 x_2)$ contains two special points y_1, y_2, and $y_1 \in x_1 y_2$. Let $\eta_i \in Q_{y_i}$ $(i = 1, 2)$ be the point in Q_{y_i} such that the subsegment from $\log_{y_i}(x_{3-i})$ to η_i has length π. We extend the geodesic $x_{3-i} y_i$ into supp(Q) in the direction of η_i until it hits a point z_i on Q. Note $y_i z_i \subset \Delta^{(1)}$. The uniqueness of geodesic implies $z_1 \in x_1 x_4 \cup x_4 x_3$ and $z_2 \in x_2 x_3 \cup x_3 x_4$. Note $z_1 z_2 = z_1 y_1 \cup y_1 y_2 \cup y_2 z_2$. The uniqueness of geodesic implies at most one of z_1, z_2 lies on $x_3 x_4$. We claim none of z_1, z_2 lies on $x_3 x_4$. Suppose the contrary holds, say, $z_2 \in x_3 x_4$. Then $z_1 \in$ interior$(x_1 x_4)$. Consider $T = (x_1, z_2, x_4)$: $z_1 \in$ interior$(x_1 x_4)$, $y_2 \in$ interior$(x_1 z_2)$ and $y_2 z_1 \cap x_1 z_2 = y_2 y_1$ is a nontrivial segment. This contradicts to Proposition 5.1. Hence $z_1 \in x_1 x_4$ and $z_2 \in x_2 x_3$. Notice $\angle_{y_i}(x_i, z_i)$ $(i = 1, 2)$ is an even angle. Proposition 5.2 applied to $T_i = (x_i, y_i, z_i)$

implies R is a right triangle. Let $Q' = (z_1, z_2, x_3, x_4)$. Lemma 10.10 and the assumption imply that $\overline{\mathrm{supp}(Q')}$ is homeomorphic to a closed disk and we can apply Corollary 10.7 to Q'.

We claim $R = (2, 3, 8)$. Suppose $R \neq (2, 3, 8)$. Then $A_0 \geq \pi/12$. Since $\angle_{y_i}(x_i, z_i)$ $(i = 1, 2)$ is an even angle, Proposition 5.2 implies $m(y_i) \geq 4$ and $\angle_{z_i}(x_i, y_i) \leq \pi/4$. It follows that $\angle_{z_1}(y_1, x_4), \angle_{z_2}(x_3, y_2) \geq 3\pi/4$ and $d(Q') \leq \pi/4$. Corollary 10.7 implies that $n(Q') \leq 3$. On the other hand, since $y_1 \in \mathrm{interior}(z_1 z_2)$ and $m(y_1) \geq 4$, there are at least 4 chambers in $\overline{\mathrm{supp}(Q')}$ incident to y_1, a contradiction.

We next claim $m(y_1) = m(y_2) = 8$. Suppose the claim is not true, say, $m(y_1) \neq 8$. Since $T_1 = (x_1, y_1, z_1)$ has an even angle at y_1, we must have $m(y_1) = 3$. Proposition 5.2 applied to T_1 implies $\angle_{z_1}(x_1, y_1) = \pi/8$. Proposition 5.2 applied to T_2 also implies $\angle_{z_2}(x_2, y_2) \leq \pi/2$. It follows that $\angle_{z_1}(x_4, y_1) = 7\pi/8$, $\angle_{z_2}(x_3, y_2) \geq \pi/2$ and $d(Q') \leq 3\pi/8$. Corollary 10.7 then implies $n(Q') \leq 9$. Now we count the chambers in $\overline{\mathrm{supp}(Q')}$ that intersect $z_1 z_2$: at least 7 chambers are incident to z_1, at least 3 are incident to y_1 and at most one of them is incident to both z_1 and y_1, at least 3 are incident to y_2 and at most one of them is incident to both y_1 and y_2. Hence there are at least 11 chambers in $\overline{\mathrm{supp}(Q')}$, contradicting to $n(Q') \leq 9$.

Notice Proposition 5.2 implies $\angle_{z_i}(x_i, y_i) \leq \pi/2$ because $\angle_{y_i}(x_i, z_i)$ $(i = 1, 2)$ is an even angle. It follows that $\angle_{z_1}(y_1, x_4), \angle_{z_2}(x_3, y_2) \geq \pi/2$. On the other hand, since $y_1, y_2 \in \mathrm{interior}(z_1 z_2)$ and $m(y_1) = m(y_2) = 8$, we have $n(Q') \geq 16$. Corollary 10.7 then implies that $\angle_{x_4}(x_3, z_1) = \angle_{x_3}(x_4, z_2) = \pi/8$ and $\angle_{z_1}(y_1, x_4) = \angle_{z_2}(x_3, y_2) = \pi/2$. In particular, $m(x_3) = m(x_4) = 8$. Corollary 10.7 applied to Q' again shows that $n(Q') = 18$. We count the chambers in $\overline{\mathrm{supp}(Q')}$ that are incident to vertices indexed by 8: at least 8 chambers are incident to each of y_1, y_2, and at least one is incident to each of x_3, x_4, for a total of 18. It follows that $\overline{\mathrm{supp}(Q')}$ is the union of these 18 chambers. Since $\angle_{z_1}(y_1, x_4) = \angle_{z_2}(x_3, y_2) = \pi/2$ and $\angle_{y_i}(x_i, z_i)$ is an even angle, Proposition 5.2 applied to T_i shows that $m(z_i) = 2$ and $z_i y_i$ is an edge. It follows that the vertices on $z_1 z_2$ are periodically indexed by 2 and 8, and there is exactly one vertex in $\mathrm{interior}(y_1 y_2)$ and it is indexed by 2. One also observes that there is exactly one vertex in the interior of each of the segments $z_1 x_4$, $z_2 x_3$ and it is indexed by 3. From these observations one sees that the union of the above 18 chambers is as shown in Figure 11. This union is not a quadrilateral, a contradiction.

□

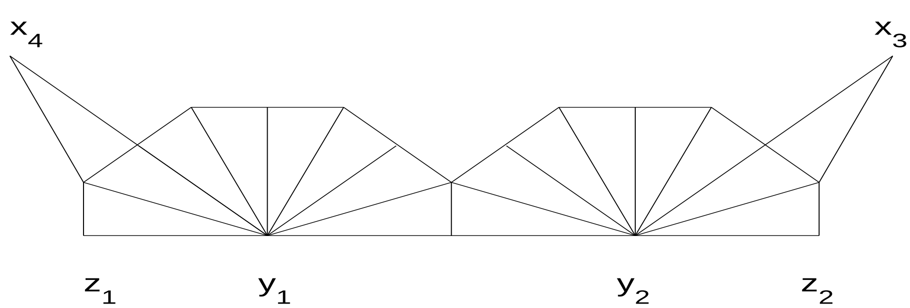

FIGURE 11.

Lemma 10.20. *Let $Q = (x_1, x_2, x_3, x_4) \subset \Delta^{(1)}$ be a quadrilateral that is homeomorphic to a circle with $d(Q) = \frac{k\pi}{24}$, $k \geq 3$. Suppose $\overline{\mathrm{supp}(Q')}$ is homeomorphic to a closed disk for every quadrilateral $Q' \subset \Delta^{(1)}$ homeomorphic to a circle with $d(Q') \leq \frac{(k-1)\pi}{24}$. If there is a special point $x \in \mathrm{supp}(Q)$, then $R = (2, 3, 8)$ and $\overline{\mathrm{supp}(Q)}$ must be as shown in Figure 12.*

Since $x \in \mathrm{supp}(Q)$ is a special point, Q_x is a circle with length $> 2\pi$. It follows that there are at least 5 edges that are contained in $\overline{\mathrm{supp}(Q)}$ and incident to x. We extend these geodesics inside

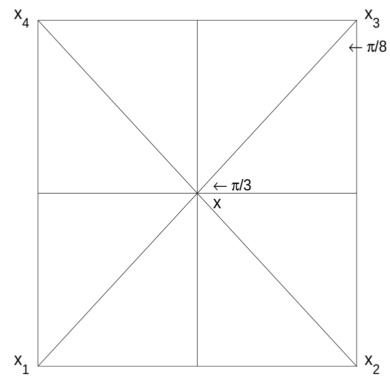

FIGURE 12. $x \in \mathrm{supp}(Q)$ is a special point of $\overline{\mathrm{supp}(Q)}$

$\mathrm{supp}(Q)$ until they hit Q. Geodesic uniqueness implies these 5 points on Q are pairwise distinct. At least one of these 5 points is different from all the $x_i's$. We denote this point by r_1 and we may assume $r_1 \in \mathrm{interior}(x_1x_2)$. Let $\xi \in Q_x$ be the initial direction of xr_1 at x. Let $\eta_2, \eta_3 \in Q_x$ be the two points in Q_x that have distance π from ξ. We extend r_1x inside $\mathrm{supp}(Q)$ in the directions η_2 and η_3 until the extensions hit Q at two points r_2 and r_3 respectively. Note $r_2, r_3 \in Q - x_1x_2$. Then exactly one of the following occurs:
(1) one of r_2, r_3 lies in x_2x_3 and the other lies in x_1x_4, but $\{r_2, r_3\} \neq \{x_3, x_4\}$;
(2) one of r_2, r_3 lies in $\mathrm{interior}(x_3x_4)$, the other lies in $\mathrm{interior}(x_2x_3)$ or in $\mathrm{interior}(x_1x_4)$;
(3) r_2, r_3 both lie in x_2x_3 or both lie in x_1x_4;
(4) $r_2, r_3 \in \mathrm{interior}(x_3x_4)$;
(5) one of r_2, r_3 lies in $\mathrm{interior}(x_3x_4)$ and the other lies in $\{x_3, x_4\}$;
(6) $\{r_2, r_3\} = \{x_3, x_4\}$.

Lemma 10.21. *Case* (1) *cannot occur.*

Proof. Suppose case (1) occurs. We may assume $r_2 \in \mathrm{interior}(x_2x_3)$ and $r_3 \in x_1x_4$. Let $T_2 = (r_2, x_2, r_1)$, $T_3 = (r_3, x_1, r_1)$. Notice the side r_1r_i of T_i contains at least two edges. By Proposition 5.2 R is a right triangle and $\angle_{r_1}(x_2, r_2), \angle_{r_1}(x_1, r_3) \leq \pi/2$. Using the triangle inequality, one further concludes that $\angle_{r_1}(x_2, r_2) = \angle_{r_1}(x_1, r_3) = \frac{\pi}{2}$. Applying Proposition 5.2 again, one sees that $R = (2, 3, 8)$ and there are two possibilities for T_i:
(a) Both T_2 and T_3 are as shown in Figure 3(b), in which case $m(x) = 2$ and T_i has angle $\pi/8$ at r_i;
(b) Both T_2 and T_3 are as shown in Figure 3(d), in which case $m(x) = 3$ and T_i has angle $\pi/8$ at r_i.

Suppose (a) occurs. Then $r_2r_3 = r_2x \cup xr_3$, and $c = (r_2, r_3, x_4, x_3)$ is a quadrilateral if $r_3 \neq x_4$ and is a triangle if $r_3 = x_4$. Since T_i has angle $\pi/8$ at r_i $(i = 2, 3)$, we have $d(c) \leq 0$, contradicting to the fact that Δ is a CAT(-1) space.

Suppose (b) holds. First assume $r_3 = x_4$. Then $Q' = (x_4, x, r_2, x_3)$ is a quadrilateral with $d(Q') < d(Q)$. The assumption implies that $\overline{\mathrm{supp}(Q')}$ is a closed disk and we can apply Corollary 10.7. Note $\angle_x(r_2, r_3) = 2\pi/3$ and $\angle_{r_2}(x, x_3) = 7\pi/8$. Hence $d(Q') \leq 5\pi/24$ and $n(Q') \leq 5$. On the other hand, there are at least 7 chambers in $\overline{\mathrm{supp}(Q')}$ that are incident to r_2, a contradiction.

Now assume $r_3 \neq x_4$. Proposition 5.1 implies $T_{2x} = \mathrm{Link}(\overline{\mathrm{supp}(T_2)}, x)$ has length π. The segment $\eta_2\eta_3 \subset \mathrm{Link}(\Delta, x)$ has length $2\pi/3$. Since $\angle_x(r_3, r_1) = \angle_x(r_2, r_1) = \pi$, $\eta_2\eta_3 \cap T_{2x}$ either equals $\{\eta_2\}$ or is a segment with length $\pi/3$. First suppose $\eta_2\eta_3 \cap T_{2x} = \{\eta_2\}$. Let y be the only vertex in $\mathrm{interior}(r_2x_2)$. Then $m(y) = 2$ and $r_3y = r_3x \cup xy$. Let $Q_1 = (r_3, y, x_3, x_4)$. Then $d(Q_1) < d(Q)$ and $\overline{\mathrm{supp}(Q_1)}$ is a closed disk. Note $d(Q_1) \leq 3\pi/8$. Hence $n(Q_1) \leq 9$. Let us count the chambers in $\overline{\mathrm{supp}(Q_1)}$: since $m(r_2) = m(r_3) = 8$ and $r_2 \in \mathrm{interior}(yx_3)$, there are at least 7 chambers incident to r_3 and at least 8 incident to r_2. Hence $n(Q_1) \geq 15$, contradiction.

Now suppose $\eta_2\eta_3 \cap T_{2x}$ is a segment with length $\pi/3$. In this case (x, y, r_3) is the boundary of a chamber. In particular, r_3y is an edge. The fact $m(y) = 2$ implies $\angle_y(r_3, x_3) = \pi$ and $T = (r_3, x_3, x_4) \subset \Delta^{(1)}$. Since $\angle_{r_3}(x, x_1) = \angle_{r_3}(x, y) = \pi/8$, we conclude $\angle_{r_3}(x_4, x_3) \geq 6\pi/8$ and $d(T) \leq 0$, which is impossible.

□

Lemma 10.22. *Case* (2) *cannot occur.*

Proof. Suppose case (2) occurs. We may assume $r_2 \in$ interior(x_2x_3) and $r_3 \in$ interior(x_3x_4). Let $T = (r_1, r_2, x_2)$, $Q_1 = (x_1, r_1, r_3, x_4)$ and $Q_2 = (x, r_2, x_3, r_3)$. Then $d(Q_i) < d(Q)$ $(i = 1, 2)$ and $\overline{\text{supp}(Q_i)}$ is a closed disk. Since $x \in$ interior(r_1r_2), Proposition 5.2 applied to T implies R is a right triangle. Note $m(x) \neq 2$: otherwise $\angle_x(r_2, r_3) = \pi$, xr_1 and xr_3 are extensions of r_2x and we have case (1), which does not occur. Proposition 5.2 then implies $R = (2, 4, 8), (2, 4, 6)$, or $(2, 3, 8)$. Assume $R = (2, 4, 8)$. Then $A_0 = \pi/8$, $m(x) = 4$ and $\angle_{r_2}(x, x_2) = \pi/8$. It follows that $\angle_{r_2}(x, x_3) = 7\pi/8$. Note $\angle_x(r_2, r_3) \geq \pi/2$ since $m(x) = 4$ and $\angle_x(r_2, r_3)$ is a nonzero even angle. Now $d(Q_2) \leq 3\pi/8$ and $n(Q_2) \leq 3$, contradicting to the fact that there are at least 7 chambers in $\overline{\text{supp}(Q_2)}$ incident to r_2. Similarly $R \neq (2, 4, 6)$. Hence $R = (2, 3, 8)$.

We claim $m(v) \neq 8$ for every vertex $v \in$ interior(r_1r_2); in particular, $m(x) = 3$. Suppose there is a vertex $v \in$ interior(r_1r_2) with $m(v) = 8$. Then Proposition 5.2 applied to T implies $\angle_{r_2}(x, x_2) = \angle_{r_1}(x, x_2) = \pi/8$ and $m(v) = 2$ or 8 for all vertices v on r_1r_2. It follows that $\angle_{r_1}(x, x_1) = \angle_{r_2}(x, x_3) = 7\pi/8$ and $m(x) = 8$ (since $m(x) \neq 2$). We count the chambers in $\overline{\text{supp}(Q_1)}$: at least 7 chambers are incident to r_1 and at least 8 are incident to x. Hence $n(Q_1) \geq 15$. Corollary 10.7 applied to Q_1 implies $\angle_{r_3}(x, x_4) \leq \pi/4$. It follows that $\angle_{r_3}(x, x_3) \geq 3\pi/4$. Since $\angle_x(r_2, r_3) \geq \pi/4$ and $\angle_{r_2}(x, x_3) = 7\pi/8$, we conclude $d(Q_2) \leq 0$, impossible.

Since $m(x) = 3$, the two vertices in r_1r_2 that are adjacent to x are indexed by 2 and 8 respectively. It follows from the last paragraph that one of the following occurs:
(a) $m(r_2) = 8$ and xr_2 is an edge;
(b) $m(r_1) = 8$ and xr_1 is an edge.
First assume (a) holds. Then Proposition 5.2 implies $\angle_{r_2}(x, x_2) = \pi/8$ or $2\pi/8$. It follows that $\angle_{r_2}(x, x_3) = 7\pi/8$ or $6\pi/8$. If $\angle_{r_2}(x, x_3) = 7\pi/8$, then $n(Q_2) \leq 5$, contradicting to the fact that there are at least 7 chambers in $\overline{\text{supp}(Q_2)}$ incident to r_2. If $\angle_{r_2}(x, x_3) = 6\pi/8$, then $n(Q_2) \leq 8$. A similar argument shows $\angle_{x_3}(r_3, r_2) < \pi/3$, in particular, $m(x_3) \neq 3$. It follows that there is a vertex $v \in$ interior(r_2x_3) with $m(v) = 3$. Now there are 6 chambers in $\overline{\text{supp}(Q_2)}$ incident to r_2 and 2 chambers incident to v but not to r_2. Hence $\overline{\text{supp}(Q_2)}$ is the union of these 8 chambers. However, this union is not a quadrilateral, a contradiction.

Now we assume (b) holds. Then $\angle_{r_1}(x, x_1) = 7\pi/8$ or $6\pi/8$. On the other hand, either $\angle_{r_3}(x, x_4) \geq \pi/2$ or $\angle_{r_3}(x, x_3) \geq \pi/2$. Assume $\angle_{r_3}(x, x_4) \geq \pi/2$ and $\angle_{r_1}(x, x_1) = 7\pi/8$. Then $n(Q_1) \leq 9$. There are 7 chambers in $\overline{\text{supp}(Q_1)}$ incident to r_1 and 2 chambers incident to x but not to r_1. It follows that $\overline{\text{supp}(Q_1)}$ is the union of these 9 chambers. However, this union is not a quadrilateral, a contradiction. Now assume $\angle_{r_3}(x, x_4) \geq \pi/2$ and $\angle_{r_1}(x, x_1) = 6\pi/8$. Then $\angle_{r_1}(x, x_2) = 2\pi/8$. Proposition 5.2 implies that $\angle_{r_2}(x, x_2) = \pi/8$. Hence $\angle_{r_2}(x, x_3) = 7\pi/8$. Also note $\angle_x(r_2, r_3) = 2\pi/3$. It follows that $n(Q_2) \leq 5$, contradicting to $\angle_{r_2}(x, x_3) = 7\pi/8$. Therefore we have $\angle_{r_3}(x, x_4) < \pi/2$. It follows that $\angle_{r_3}(x, x_3) \geq 5\pi/8$. Since $\angle_{r_2}(x, x_2) \leq \pi/2$, we have $\angle_{r_2}(x, x_3) \geq \pi/2$. Consequently $d(Q_2) \leq \pi/12$ and $n(Q_2) \leq 2$. It follows that $\overline{\text{supp}(Q_1)}$ is the union of the two chambers incident to x. In particular, $m(r_3) = 2$, contradicting to $\angle_{r_3}(x, x_3) \geq 5\pi/8$.

□

Lemma 10.23. *Case* (3) *cannot occur.*

Proof. Suppose case (3) occurs. We may assume $r_2, r_3 \in x_2x_3$ and $r_2 \in \text{interior}(r_3x_2)$. Let $T = (r_1, r_3, x_2)$. Since $r_2 \in \text{interior}(r_3x_2)$ and $x \in \text{interior}(r_3r_1)$, Proposition 5.1 implies $\angle_x(r_1, r_2) < \pi$, contradicting to the definition of r_2.

□

Lemma 10.24. *Case* (4) *cannot occur.*

Proof. Suppose case (4) occurs. We may assume $r_3 \in \text{interior}(x_4r_2)$. Let $T = (x, r_2, r_3)$, $Q_1 = (x_1, r_1, r_3, x_4)$ and $Q_2 = (x_2, r_1, r_2, x_3)$. Then $d(Q_i) < d(Q)$ $(i = 1, 2)$ and $\overline{\text{supp}(Q_i)}$ is a closed disk. Note T has an even angle at x. Proposition 5.2 implies R is a right triangle. We claim $R = (2, 3, 8)$, that is, $R \neq (2, 8, 8)$, $(2, 6, 8)$, $(2, 6, 6)$, $(2, 4, 8)$, $(2, 4, 6)$. We only show $R \neq (2, 4, 8)$, the other cases can be handled similarly. Assume $R = (2, 4, 8)$. Then $A_0 = \pi/8$. At least one of $\angle_{r_1}(x, x_1), \angle_{r_1}(x, x_2) \geq \pi/2$. We may assume $\angle_{r_1}(x, x_2) \geq \pi/2$. Proposition 5.2 applied to T shows $\angle_{r_2}(x, r_3) \leq \pi/4$, which implies $\angle_{r_2}(x, x_3) \geq 3\pi/4$. It follows that $d(Q_2) \leq \pi/2$ and $n(Q_2) \leq 4$. On the other hand, since $m(x) \geq 4$ and $\angle_{r_2}(x, x_3) \geq 3\pi/4$, there are at least 4 chambers in $\overline{\text{supp}(Q_2)}$ incident to x, and at least 2 chambers incident to r_2 but not to x. Hence $n(Q_2) \geq 6$, contradicting to $n(Q_2) \leq 4$.

We claim $m(x) = 8$. Assume $m(x) \neq 8$. Since T has an even angle at x, $m(x) = 3$. Proposition 5.2 implies $\angle_{r_2}(r_3, x) = \angle_{r_3}(r_2, x) = \pi/8$. Then $\angle_{r_2}(x_3, x) = \angle_{r_3}(x_4, x) = 7\pi/8$. At least one of $\angle_{r_1}(x, x_1)$, $\angle_{r_1}(x_2, x)$ is $\geq \pi/2$. We may assume $\angle_{r_1}(x_2, x) \geq \pi/2$. Then $d(Q_2) \leq 3\pi/8$ and $n(Q_2) \leq 9$. We count the chambers in $\overline{\text{supp}(Q_2)}$: at least 7 chambers are incident to r_2 and at least 2 chambers are incident to x but not to r_2. Hence $\overline{\text{supp}(Q_2)}$ must be the union of these 9 chambers. But this union is not a quadrilateral. Contradiction.

Assume T is as shown in Figure 3(d). We may assume $m(r_3) = 2$ and $\angle_{r_2}(x, r_3) = \pi/8$. Since $m(x) = 8$, by counting the chambers in $\overline{\text{supp}(Q_2)}$ incident to x and r_2, we see $n(Q_2) \geq 15$. Corollary 10.7 then implies $\angle_{r_1}(x, x_2) \leq \pi/4$. It follows that $m(r_1) = 8$ and $\angle_{r_1}(x, x_1) \geq 3\pi/4$. Note $\angle_{r_3}(x_4, x) = \pi/2$. Corollary 10.7 applied to Q_1 implies $n(Q_1) \leq 12$. On the other hand, by counting the chambers in $\overline{\text{supp}(Q_1)}$ incident to x and r_1 we conclude $n(Q_1) \geq 14$, a contradiction.

Assume T is as shown in Figure 3(f). Then $\angle_{r_2}(r_3, x) = \angle_{r_3}(r_2, x) = \pi/3$. At least one of $\angle_{r_1}(x, x_1)$, $\angle_{r_1}(x_2, x)$ is $\geq \pi/2$. We may assume $\angle_{r_1}(x_2, x) \geq \pi/2$. Corollary 10.7 implies $n(Q_2) \leq 14$. It follows that $m(v) \neq 8$ for all vertices $v \in \text{interior}(xr_1)$. Notice the vertices on r_1r_2 are periodically indexed by $8, 3, 2, 3$. It implies that there are at most three vertices in $\text{interior}(xr_1)$. Assume there is no or exactly two vertices in $\text{interior}(xr_1)$. Then $\angle_{r_1}(x_2, x) = 2\pi/3$. Corollary 10.7 implies $n(Q_2) \leq 10$. By counting the chambers in $\overline{\text{supp}(Q_2)}$ intersecting r_1r_2 we see there cannot be exactly two vertices in $\text{interior}(xr_1)$ and hence xr_1 is an edge. In this case, there are at least 8 chambers in $\overline{\text{supp}(Q_2)}$ incident to x, 1 incident to r_2 but not to x and 1 incident to r_1 but not to x. It follows that $\overline{\text{supp}(Q_2)}$ is the union of these 10 chambers. However, this union is not a quadrilateral, a contradiction. The counting argument also shows that it is impossible to have $m(r_1) = 8$ (with $\angle_{r_1}(x, x_2) \geq \pi/2$). Hence there is exactly one vertex $v \in \text{interior}(xr_1)$ and $m(v) = 3$, $m(r_1) = 2$. Since $n(Q_2) \leq 14$ and $x \in \text{interior}(r_1r_2)$ with $m(x) = 8$, $m(z) \neq 8$ for every vertex $z \in \overline{\text{supp}(Q_2)}$ different from x and the corners. Let C be the chamber in $\overline{\text{supp}(Q_2)}$ containing vr_1. C has a vertex $v' \neq x$ with $m(v') = 8$. Notice $v' \in r_1x_2$. It follows that $v' = x_2$. Consider the three chambers in $\overline{\text{supp}(Q_2)}$ that are incident to v. Two of them are incident to x_2. It follows that $\angle_{x_2}(r_1, x_3) \geq \pi/4$. Corollary 10.7 then implies $n(Q_2) \leq 11$. One observes that there are at least 11 chambers in $\overline{\text{supp}(Q_2)}$ intersecting r_1r_2, but their union is not a quadrilateral, a contradiction.

For the remaining cases, $\angle_{r_2}(r_3, x)$ and $\angle_{r_3}(r_2, x) \leq \pi/4$ hold. In particular, $m(r_2) = m(r_3) = 8$ and Corollary 10.7 implies $n(Q_1) \leq 12$ or $n(Q_2) \leq 12$. We may assume $n(Q_2) \leq 12$. On the other hand, $n(Q_2) \geq 14$ since there are at least 8 chambers in $\overline{\text{supp}(Q_2)}$ incident to x and at least 6 incident to r_2. A contradiction.

□

Lemma 10.25. *Case* (5) *cannot occur.*

Proof. Suppose case (5) does occur. We may assume $r_2 = x_3$ and $r_3 \in \text{interior}(x_3x_4)$. Let $T_1 = (r_1, x_2, x_3)$, $T_2 = (x, x_3, r_3)$ and $Q' = (x_1, r_1, r_3, x_4)$. Then $d(Q') < d(Q)$ and $\overline{\text{supp}(Q')}$ is a closed disk. Note T_2 has an even angle at x. It follows that R is a right triangle and $m(x) \neq 2$. We claim $R = (2,3,8)$. Suppose $R \neq (2,3,8)$. Then $A_0 \leq \pi/12$. Proposition 5.2 applied to T_1 and T_2 implies $\angle_{r_3}(x, x_3), \angle_{r_1}(x, x_2) \leq \pi/4$. It follows that $\angle_{r_1}(x, x_1), \angle_{r_3}(x, x_4) \geq 3\pi/4$ and $d(Q') \leq \pi/4$. Corollary 10.7 then implies $n(Q') \leq 3$. On the other hand, since $R \neq (2,3,8)$ is a right triangle and $m(x) \neq 2$, we have $m(x) \geq 4$. It follows that $n(Q') \geq 4$, contradicting to $n(Q') \leq 3$.

Assume $m(x) = 3$. Recall T_2 has an even angle at x. Proposition 5.2 applied to T_2 and T_1 implies $\angle_{r_3}(x, x_3) = \pi/8$, $\angle_{r_1}(x, x_2) \leq \pi/2$. It follows that $\angle_{r_3}(x_4, x) = 7\pi/8$ and $\angle_{r_1}(x_1, x) \geq \pi/2$. Consequently, $d(Q') \leq 3\pi/8$ and $n(Q') \leq 9$. There are at least 7 chambers in $\overline{\text{supp}(Q')}$ incident to r_3 and at least 2 incident to x but not to r_3. Hence $\overline{\text{supp}(Q')}$ must be the union of these 9 chambers. However, this union is not a quadrilateral, a contradiction. Therefore $m(x) = 8$. Then Proposition 5.2 implies $\angle_{r_1}(x, x_2) = \pi/8$, $\angle_{r_3}(x, x_3) \leq \pi/2$. In particular, $m(r_1) = 8$. It follows that $\angle_{r_1}(x_1, x) = 7\pi/8$, $d(Q') \leq 3\pi/8$ and $n(Q') \leq 9$. On the other hand, there are 8 chambers in $\overline{\text{supp}(Q')}$ incident to x and 7 incident to r_1. Hence $n(Q') \geq 15$, a contradiction.

□

Lemma 10.26. *Suppose case* (6) *occurs. Then* $R = (2,3,8)$ *and* $\overline{\textit{supp}(Q)}$ *must be as shown in Figure 12.*

Proof. We may assume $r_2 = x_3$ and $r_3 = x_4$. Let $T_1 = (x_1, r_1, x_4)$, $T_2 = (x_2, r_1, x_3)$ and $T_3 = (x, x_3, x_4)$. Note T_3 has an even angle at x. It follows that R is a right triangle and $m(x) \geq 3$. We claim $R = (2,3,8)$. Suppose $R \neq (2,3,8)$. Then Proposition 5.2 applied to T_i $(i = 1,2)$ implies $\angle_{r_1}(x, x_i) \leq \pi/6$. By the triangle inequality $\angle_{r_1}(x_1, x_2) \leq \pi/3$, contradicting to $r_1 \in x_1x_2$.

We claim $m(v) \neq 8$ for every vertex v in the interior of r_1x_3 or r_1x_4. Suppose $m(v) = 8$ for some vertex $v \in \text{interior}(r_1x_3)$. Then Proposition 5.2 applied to T_2 implies $\angle_{r_1}(x_2, x) = \pi/8$. It follows that $\angle_{r_1}(x, x_1) \geq 7\pi/8$ and $d(T_1) < 0$, which is impossible. In particular, $m(x) \neq 8$ and hence $m(x) = 3$. Since T_3 has an even angle at x, Proposition 5.2 implies $\overline{\text{supp}(T_3)}$ is as shown in Figure 3(e). On the other hand, Proposition 5.2 applied to T_1 and T_2 shows $\angle_{r_1}(x, x_1) \leq \pi/2$, $\angle_{r_1}(x, x_2) \leq \pi/2$. Triangle inequality further implies $\angle_{r_1}(x, x_1) = \angle_{r_1}(x, x_2) = \pi/2$. Now Proposition 5.2 applied to T_1 and T_2 again implies that $\overline{\text{supp}(T_i)}$ $(i = 1,2)$ must be as shown in Figure 3(d), with $m(x) = 3$. The lemma is proved.

□

The proof of Proposition 5.3 is now complete.

References

[BB] W. Ballmann, S. Buyalo, *Nonpositively curved metrics on* 2*-polyhedra,* Math. Z. **222** (1996), no. 1, 97–134.

[Bea] A. Beardon, *The geometry of discrete groups,* Graduate Texts in Mathematics, **91**. Springer-Verlag, New York, 1995.

[Ben] N. Benakli, *Polyedres hyperboliques, passage du local au global,* Thesis, Universite Paris Sud, 1992.

[Bo1] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidit de Mostow,* Geom. Funct. Anal. **7** (1997), no. 2, 245–268.

[Bo2] M. Bourdon, *Sur les immeubles fuchsiens et leur type de quasi-isomtrie,* Ergodic Theory Dynam. Systems **20** (2000), no. 2, 343–364.

[BP1] M. Bourdon, H. Pajot, *Poincar inequalities and quasiconformal structure on the boundary of some hyperbolic buildings,* Proc. Amer. Math. Soc. **127** (1999), no. 8, 2315–2324.

[BP2] M. Bourdon, H. Pajot, *Rigidity of quasi-isometries for some hyperbolic buildings,* Comment. Math. Helv. **75** (2000), no. 4, 701–736.

[BP3] M. Bourdon, H. Pajot, *Quasi-conformal geometry and hyperbolic geometry,* in "Rigidity in dynamics and geometry", (Cambridge, 2000), 1–17, Springer, Berlin, 2002.

[BH] M. Bridson, A. Haefliger, *Metric spaces of nonpositive curvature,* Grundlehren **319**, Springer-Verlag, Berlin(1999).

[F] H. Federer, *Geometric measure theory,* Grundlehren **153**, Springer-Verlag, New York, 1969.

[FH] W. Feit, G. Higman, *The nonexistence of certain generalized polygons,* J. Algebra **1** (1964) 114–131.

[G] M. Gromov, *Hyperbolic Groups,* in "Essays in Group Theory" (ed. S.Gersten) M.S.R.I. Publications No.**8**, Springer-Verlag(1987) 75–263.

[GP] D. Gaboriau, F. Paulin, *Sur les immeubles hyperboliques,* Geom. Dedicata **88** (2001), no. 1-3, 153–197.

[H] J. Heinonen, *Lectures on analysis on metric spaces,* Universitext. Springer-Verlag, New York, 2001.

[HK] J. Heinonen, P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry,* Acta Math. **181** (1998), no. 1, 1–61.

[KK] B. Kleiner, M. Kapovich, *Hyperbolic groups with low-dimensional boundary,* Ann. Sci. cole Norm. Sup. (4) **33** (2000), no. 5, 647–669.

[KL] B. Kleiner, B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings,* C. R. Acad. Sci. Paris Sr. I Math. **324** (1997), no. 6, 639–643.

[M] B. Maskit, *Kleinian Groups*, Springer, 1987.

[P] P. Pansu, *Mtriques de Carnot-Carathodory et quasiisomtries des espaces symtriques de rang un,* Ann. of Math. (2) **129** (1989), no. 1, 1–60.

[R] M. Ronan, *Lectures on buildings,* Perspectives in Mathematics **7**. Academic Press, Inc., Boston, MA, 1989.

[RT] M. Ronan,J. Tits, *Building buildings,* Math. Ann. **278** (1987), no. 1-4, 291–306.

[S1] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions,* Inst. Hautes tudes Sci. Publ. Math. No. **50** (1979), 171–202.

[S2] D. Sullivan, *Discrete conformal groups and measurable dynamics,* Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 1, 57–73.

[T] J. Tits, *Buildings of spherical type and finite BN-pairs,* Lecture Notes in Mathematics, Vol. **386**. Springer-Verlag, Berlin-New York, 1974.

[X] X. Xie, *The Tits boundary of a* $CAT(0)$ *2-complex,* to appear in Transactions of AMS.